STANLEY SEQUENCES WITH ODD CHARACTER

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Abstract. Given a set of integers containing no 3-term arithmetic progressions, one constructs a Stanley sequence by choosing integers greedily without forming such a progression. Independent Stanley sequences are a "well-structured" class of Stanley sequences with two main parameters: the character $\lambda(A)$ and the repeat factor $\rho(A)$. Rolnick conjectured that for every $\lambda \in \mathbb{N}_0 \backslash \{1, 3, 5, 9, 11, 15\}$, there exists an independent Stanley sequence $S(A)$ such that $\lambda(A) = \lambda$. This paper demonstrates that $\lambda(A) \notin \{1, 3, 5, 9, 11, 15\}$ for any independent Stanley sequence $S(A)$.

1. Introduction

Let $\mathbb{N}_0$ denote the set of non-negative integers. A subset of $\mathbb{N}_0$ is called ℓ-free if it contains no ℓ-term arithmetic progression. We will frequently abbreviate "arithmetic progression" by AP. We say a subset, or sequence of elements, of $\mathbb{N}_0$ is free of arithmetic progressions if it is 3-free. In 1978, Odlyzko and Stanley [2] used a greedy algorithm (see Definition 1.1) to produce arithmetic progression free sequences. Their algorithm produced sequences with two distinct growth rates – those which are highly structured (Type I) and those which are seemingly random (Type II). These classes of Stanley sequences will be more precisely defined in Conjecture 1.3.

Definition 1.1. Given a finite 3-free set $A = \{a_0, \dots, a_n\} \subset \mathbb{N}_0$, the *Stanley sequence generated by* A is the infinite sequence $S(A) = \{a_0, a_1, \dots\}$ defined by the following recursion. If $k \geq n$ and $a_0 < \dots < a_k$ have been defined, let a_{k+1} be the smallest integer $a > a_k$ such that $\{a_0, \dots, a_k\} \cup \{a\}$ is 3-free. Though formally one writes $S(\{a_0, \dots, a_n\})$, we will frequently use the notation $S(a_0, \dots, a_n)$ instead.

Remark 1.2. Without loss of generality, we may assume that every Stanley sequence begins with 0.

In Rolnick's investigation of Stanley sequences [3], he made the following conjecture about the growth rate of the two types of Stanley sequences.

Conjecture 1.3. *Let $S(A) = (a_n)$ be a Stanley sequence. Then, for all n large enough, one of the following two patterns of growth is satisfied:*

- *Type I:* $\alpha/2 \leq \liminf a_n/n^{\log_2(3)} \leq \limsup a_n/n^{\log_2(3)} \leq \alpha$, *or*
- *Type II:* $a_n = \Theta(n^2/\ln(n))$.

Though Type II Stanley sequences are mysterious, a great deal of progress has been made in classifying Type I Stanley sequences [1]. In [3], Rolnick introduced the concept of the *independent Stanley sequence*. These Stanley sequences follow Type I growth and are defined as follows:

Definition 1.4. A Stanley sequence $S(A) = (a_n)$ is *independent* if there exists constants $\lambda = \lambda(A)$ and $\kappa = \kappa(A)$ such that for all $k \geq \kappa$ and $0 \leq i < 2^k$, we have

- $a_{2^k+i} = a_{2^k} + a_i$
- $a_{2^k} = 2a_{2^k-1} - \lambda + 1$.

The constant λ is called the *character*, and it is easy to show that $\lambda \geq 0$ for all independent Stanley sequences. If κ is taken as small as possible, then a_{2^κ} is called the *repeat factor*. Informally, κ is the point at which the sequence begins its repetitive behavior. Rolnick and Venkataramana proved that every sufficiently large integer ρ is the repeat factor of some independent Stanley sequence [4].

Rolnick also made a table [3] of independent Stanley sequences with various characters $\lambda \geq 0$. He found Stanley sequences with every character up to 75 with the exception of those in the set $\{1, 3, 5, 9, 11, 15\}$. He proved that, for an independent Stanley sequence $S(A)$, $\lambda(A) \neq 1, 3$ [3, Proposition 2.12]. In light of his observations, he made the following conjecture:

Conjecture 1.5 (Conjecture 2.15, [3]). *The range of the character function is exactly the set of non-negative integers λ that are not in the set $\{1, 3, 5, 9, 11, 15\}$.*

In recent work, Sawhney [5] has shown that a positive density of even integers appear as characters of independent Stanley sequences. Analyzing the character of an independent Stanley sequence is closely related to another feature of a Stanley sequence which we introduce now.

Definition 1.6. Given a Stanley sequence $S(A)$, we define the omitted set $O(A)$ to be the set of nonnegative integers that are neither in $S(A)$ nor are covered by $S(A)$. For $O(A) \neq \emptyset$, we let $\omega(A)$ denote the largest element of $O(A)$.

Remark 1.7. The only Stanley sequence $S(A)$ where $O(A) = \emptyset$ is $S(0)$.

Using this definition, one can show the following lemma.

Lemma 1.8 (Lemma 2.13, [3]). *If $S(A)$ is independent, then $\omega(A) < \lambda(A)$.*

Since $\max(A) > \omega(A)$, the following corollary easily follows.

Corollary 1.9 (Corollary 2.14, [3]). *At most finitely many independent Stanley sequences exist with a given character λ.*

Using this corollary, one can show that there are no independent Stanley sequences of a given character λ by classifying every Stanley sequence with $\omega < \lambda$. One can utilize this technique to prove that $\lambda \neq 1, 3$ because every Stanley sequence with $\omega(A) < 3$ is independent with $\lambda(A) \neq 1, 3$. Unfortunately, this argument does not work for $\lambda = 5$ because the Stanley sequence $S(0, 4)$ does not *appear* to be independent and experimentally exhibits Type II growth. Though no Stanley sequence, including $S(0, 4)$, has been proven to follow Type II growth, we will prove that no independent Stanley sequence has character $\lambda = 1, 3, 5, 9, 11, 15$ by showing sequences such as $S(0, 4)$ cannot be independent *and* have certain characters.

Theorem 1.10. *Let $S(A)$ be an independent Stanley sequence where A is a finite 3-free subset of $\mathbb{N}_0$. Then $\lambda(A) \notin \{1, 3, 5, 9, 11, 15\}$.*

2. Modular sequences

In order to prove our main result, we will use the theory of modular sequences developed in [1] and more recently studied in [6]. Modular sequences are a class of Stanley sequences of Type I which contains all independent Stanley sequences as a strictly smaller subset.

Definition 2.1. Let A be a set of integers and z be an integer. We say that z is *covered* by A if there exist $x, y \in A$ such that $x < y$ and $2y - x = z$. We frequently say that z is covered by x and y.

Suppose that N is a positive integer. If $x, y, z \in \{0, \ldots, N-1\}$ and $x \neq y$, we say they form an arithmetic progression modulo N, or a *mod-AP* if $2y - x \equiv z \pmod{N}$.

Suppose again that N is a positive integer and $A \subseteq \{0, \ldots, N-1\}$. Then, we say that z is *covered by A modulo N*, or *mod-covered*, if there exist $x, y \in A$ with $x < y$ such that x, y, z form an arithmetic progression modulo N.

Definition 2.2. Fix a positive integer $N \geq 1$. Suppose the set $A \subset \{0, \ldots, N-1\}$ containing 0 is 3-free modulo N, and all $x \in \{0, \ldots, N-1\} \backslash A$ are covered by A modulo N. Then A is said to be a *modular set modulo N* and $S(A)$ is said to be a *modular Stanley sequence modulo N*.

Observe that the modulus N of a modular Stanley sequence is analagous to the repeat factor ρ of an independent Stanley sequence. One can make this statement more precise in the following proposition:

Proposition 2.3 (Proposition 2.3, [1]). *Suppose A is a finite subset of $\mathbb{N}_0$ and suppose $S(A)$ is an independent Stanley sequence with repeat factor ρ. Then $S(A)$ is a modular Stanley sequence modulo $3^\ell \cdot \rho$ for some integer $\ell \geq 0$.*

Remark 2.4. One can show that the modulus of a modular Stanley sequence is well-defined up to a power of 3.

Definitions made about independent Stanley sequences generalize nicely to modular Stanley sequences.

Definition 2.5. Suppose that A is a modular set modulo N. Define $\lambda(A) = 2 \cdot \max(A) - N + 1$ and define $\omega(A)$ to be the largest element $x \in \{0, 1, \ldots, N-1\} \backslash A$ such that x is covered by A modulo N but x is *not* covered by A.

The definitions of λ and ω coincide with the corresponding definitions for an independent Stanley sequence when $S(A)$ is an independent Stanley sequence.

Remark 2.6. Throughout this paper, we will repeatedly use the fact that, for a modular set A modulo N, every element $x \in \{0, 1, \ldots, N-1\} \backslash A$, such that $x > \omega(A)$, is *covered* by A (and not merely mod-covered by A).

3. Proof of Main Result

Theorem 3.1. *If A is a modular set modulo $N \in \mathbb{N}$, then $\lambda(A) \notin \{1, 3, 5, 9, 11, 15\}$.*

Observe that this result implies Theorem 1.10 since every independent Stanley sequence is a modular Stanley sequence.

The proof of Theorem 3.1 has been broken up into several more manageable results including Lemma 3.3, Lemma 3.5, Proposition 3.7, Proposition 3.10, Proposition 3.15, and Proposition 3.22. The proofs of Lemmas 3.3 and 3.5 and Proposition 3.7 are more detailed in order to give the reader better guidance in understanding the various proof techniques. The later lemmas and propositions omit some details for brevity.

Lemma 3.2, though simple, will prove invaluable.

Lemma 3.2. *Suppose that $A = \{a_0, \ldots, a_n\}$ with $0 = a_0 < \cdots < a_n$ is a modular set modulo N for some $N \in \mathbb{N}$. If $a_k > \omega(A)$, then $A = S(a_0, \ldots, a_k) \cap \{0, 1, \ldots, N-1\}$ and $S(A) = S(a_0, \ldots, a_k)$.*

Proof. If $x \in \mathbb{N}$ with $x \leq a_k$ then $x \in A$ if and only if $x \in \{a_0, \ldots, a_k\}$. Therefore $S(a_0, \ldots, a_k) \cap \{0, 1, \ldots, a_k\} = A \cap \{a_0, \ldots, a_k\}$. Now we proceed by induction. Suppose that $S(a_0, \ldots, a_k) \cap \{0, 1, \ldots, a_m\} = A \cap \{0, 1, \ldots, a_m\}$ for some $k \leq m < n$. If $z \in \mathbb{N}$ and $a_m < z < a_{m+1}$ then $z \notin A$ and $z > \omega(A)$. Therefore, there exist $a_i, a_j \in A$ with $a_i < a_j$ such that a_i, a_j, z form an AP. Since $i, j \leq m$ we see that $a_i, a_j \in S(a_0, \ldots, a_k)$ and therefore $z \notin S(a_0, \ldots, a_k)$. The greedy algorithm then dictates that $a_{m+1} \in S(a_0, \ldots, a_k)$ and $S(a_0, \ldots, a_k) \cap \{0, 1, \ldots, a_{m+1}\} = A \cap \{0, 1, \ldots, a_{m+1}\}$. By induction we have shown that $S(a_0, \ldots, a_k) \cap \{0, 1, \ldots, N-1\} = A$ and $S(a_0, \ldots, a_k) = S(A)$. □

We begin by proving a few simple lemmas. In all of these lemma, the character being investigated is odd, thus the modulus is required to be even (see Definition 2.5). Therefore, we will only consider modular sets with modulus $2N$ for some $N \in \mathbb{N}$.

3.1. **Characters $\lambda = 1, 3$.**

Lemma 3.3. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 1$.*

Proof. Let A be a modular set with modulus $2N$ where $N \in \mathbb{N}$ and $\lambda(A) = 1$. Using the definition of λ, one finds that $\max(A) = N$. Contradiction. Every modular set contains 0; therefore, A contains the mod $2N$ arithmetic progression $0, N, 0$. □

Remark 3.4. The proof of Lemma 3.3 relied on the fact that if a modular set has modulus $2N$ and $x \in A$ then $x + N \pmod{2N} \notin A$. We will use this fact repeatedly throughout the proofs of the following statements.

Lemma 3.5. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 3$.*

Proof. Let A be a modular set with modulus $2N$ where $N \in \mathbb{N}$ and $\lambda(A) = 3$. One deduces that $\max(A) = N + 1$ from the definition of λ and $1, N \notin A$ by Remark 3.4. Since $1 \notin A$, it must be mod-covered by A by the definition of a modular set. That is, there exist $x, y \in A$ with $x < y$ such that $2y - x \equiv 1 \pmod{2N}$. Since $0 < y < 2N$, one deduces that $2y - x = 1$ or $2N + 1$. Since $y > 1$ we also know that $2y - x \geq y + 1 > 1$ and therefore $2y - x \neq 1$. If $y < N$, then $2y - x < 2N - x < 2N + 1$. Therefore, if $2y - x = 2N + 1$, then $y \geq N$ and $y = N + 1 = \max(A)$ necessarily. Finally, if $y = N + 1$, then $2y - (2N + 1) = x = 1 \in A$, a contradiction. □

Lemmas 3.3 and 3.5 were proven by Rolnick [3] in the case of independent Stanley sequences. We have proved these statements here as a warm-up for the upcoming more involved proofs.

Remark 3.6. In [1], an operation was introduced that allows one to combine the modular sets. If A and B are modular sets modulo N and M then $A \otimes B := A + N \cdot B$ is a modular set modulo NM with $\lambda(A \otimes B) = \lambda(A) + N \cdot \lambda(B)$. Through the following proofs, we will assume that N is "large." Let $\{0, 1\}$ be the modular set of modulus 3 with character 0. If A is a modular set modulo $2N$ then $A \otimes \{0, 1\}$ is a modular set modulo $3 \cdot 2N$ with the same character λ. Thus, if we show that there is no modular set A with odd character λ of modulus $2N$ where $N > 100$ (or any fixed number), then we have shown there exist no modular sets A of character λ.

3.2. **Character $\lambda = 5$.**

Proposition 3.7. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 5$.*

We will break the proof of Proposition 3.7 into Lemmas 3.8 and 3.9.

Lemma 3.8. *Let A be a modular set modulo $2N$ with $\lambda(A) = 5$. Then, $N + 1 \notin A$.*

Proof. Suppose that $N + 1 \in A$. Observe that $\max(A) = N + 2$ and 2 is mod-covered by $0, N+1, 2$ and $1, N \notin A$. Since $1 \notin A$, there exist $x, y \in A$ with $x < y$ such that $x, y, 1$ form a mod-AP. Since $y > 0$, we deduce that $2y - x = 2N + 1$ which further implies that $y = N + 2$ and $x = 3$. Since we now have $3 \in A$, we see that $A = S(0, 3, 5) \cap \{0, 1, \ldots, 2N-1\}$ by Lemma 3.2 and therefore $S(A) = S(0, 3, 5)$. A quick computation shows that $S(0, 3, 5) = S(B)$ where $B = \{0, 3, 5, 8\}$, a modular set modulo 9 with character $\lambda(B) = 8$. Therefore, $\lambda(A) = 8$ since $S(A) = S(B)$. Contradiction. $\square$

Lemma 3.9. *There does not exist a modular set A of modulus $2N$ and $\lambda(A) = 5$ with $N + 1 \notin A$.*

Proof. Let A be a modular set modulo $2N$ with $\lambda(A) = 5$ and $N + 1 \notin A$. Observe that $\max(A) = N + 2$ and $2 \notin A$. Since $2 \notin A$, there exist $x, y \in A$ that mod-cover 2. A quick computation shows that we require $x = 0$ and $y = 1$. Thus $1 \in A$ and we see that 3 is mod-covered by $1, N + 2, 3$ and 4 is mod-covered by $0, N + 2, 4$. Is $5 \in A$? If not, then there exist $x, y \in A$ that cover 5 mod $2N$ since $5 > \omega(A)$. This is impossible and thus $5 \in A$. Hence, $S(A) = S(0, 1, 5) = \{0, 1, 5, 6, 8, 13, \ldots\}$ by Lemma 3.2.

Since N is "big," we know that $2N-1, 2N-2, 2N-3, 2N-4, \ldots, N+3 \notin A$. Hence, these numbers are mod-covered by A and are in fact *covered* by A since $\omega(A) < 5$. We see that $2N - 1$ is covered by $5, N+2$ and $2N - 2$ is covered by $6, N + 2$. However, we can only cover $2N - 3$ by $1, N - 1$ which implies $N - 1 \in A$. We see $2N - 4$ is covered by $8, N + 2$. We know $2N-5 \notin A$ and is therefore covered by $x, y \in A$ with $x < y$. We see that $y \neq N+2$ otherwise $9 \in A$, a contradiction. We also see that $y \neq N - 1$ otherwise $3 \in A$, a contradiction. We could cover $2N - 5$ by $1, N - 2$, but this is a contradiction because then A contains the mod-AP $N - 2, 0, N + 2$. Hence, $y < N - 2$. However, $2N - 5 = 2y - x \leq 2(N - 3) + x$, a contradiction.

Therefore, there does not exist a modular set A of modulus $2N$ with $\lambda(A) = 5$ with $N + 1 \notin A$. $\square$

The techniques from Lemmas 3.8 and 3.9 will be used repeatedly in the following propositions and lemmas.

3.3. **Character $\lambda = 9$.**

Proposition 3.10. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 9$.*

We break the proof of Proposition 3.10 into Lemmas 3.11, 3.12, 3.13, and 3.14. Through case by case analysis, we will eliminate all possible sets A.

Lemma 3.11. *Let A be a modular set modulo $2N$ with $\lambda(A) = 9$. Then $N + 3 \notin A$.*

Proof. Suppose that $N+3 \in A$. Since $N+4 = \max(A)$, we see that $N+2 \notin A$ otherwise A would contain the AP $N+2, N+3, N+4$. We also see that $3, 4 \notin A$ and 6 is mod-covered by $0, N+3$ and 8 is mod-covered by $0, N+4$. The only way to mod-cover 3 is with $5, N+4$ and every valid way to mod-cover 4 requires 2; hence, $2, 3 \in A$. Since $0, 2 \in A$, we see that $N+1 \notin A$. There is no way to mod-cover 7, so $7 \in A$. We see that 9 is covered by $5, 7$ and 10 is covered by $0, 5$. However, 11 cannot be covered, so $11 \in A$ and thus we have deduced that $S(A) = S(0, 2, 5, 7, 11) = \{0, 2, 5, 7, 11, 13, 16, 18, 28, \dots\}$.

Now we examine how $2N-1, 2N-2, \dots$ are covered by A. We see that $2N-1$ is covered by $7, N+3$ but the only way to cover $2N-2$ is with $0, N-1$. Hence, $N-1 \in A$. Similar analysis shows that $2N-3$ is covered by $11, N+4$, the element $2N-4$ is covered by $2, N-1$, and the element $2N-5$ is covered by $11, N+3$. However, $2N-6$ cannot be covered by $x < y$ using $y = N+4, N+3$ or $N-1$. We see that $y = N-3$ and $y = N-2$ are the only possible remaining choices. However, $N-3$ cannot be in A, otherwise A contains the AP $N-3, 0, N+3$. Therefore, $y = N-2$ and $x = 2$ which implies that $N-2 \in A$.

Further analysis shows that $2N-7, \dots, 2N-13$ are covered by A. However, $2N-14$ cannot be covered by $x, y \in A$ with $y \in \{N-2, N-1, N+3, N+4\}$. Therefore, $y \in \{N-7, N-6, N-5, N-4, N-3\}$. However, $N-3, N-4 \notin A$ by Remark 3.4. Furthermore, $N-5 \notin A$ otherwise A would contain the AP $N-5, N-1, N+3$. Similarly, one deduces that $N-6, N-7 \notin A$. Contradiction. □

Lemma 3.12. *Let A be a modular set modulo $2N$ with $\lambda(A) = 9$ and $N+3 \notin A$. Then $N+1 \notin A$.*

Proof. Suppose that $N+1 \in A$. We see that 2 and 8 are mod-covered by A and that $1, 4 \notin A$. The only way to mod-cover 4 is with $0, N+2$; therefore, $N+2 \in A$. Observe that $3 \notin A$ otherwise A would contain the mod-AP $N+2, 3, N+4$. Therefore, the only way to mod-cover 1 is with $7, N+4$ which implies $7 \in A$. The only way to mod-cover 3 is with $5, N+4$ which implies $5 \in A$. Since $5, 7 \in A$, we see that $6 \notin A$ yet unfortunately 6 cannot be mod-covered by A. Contradiction. □

Lemma 3.13. *Let A be a modular set modulo $2N$ with $\lambda(A) = 9$ and $N+1, N+3 \notin A$. Then $N+2 \notin A$.*

Proof. Suppose that $N+2 \in A$. We see that $4, 8$ are mod-covered by A and $2 \notin A$. Also observe that $3 \notin A$ since otherwise A would contain the mod-AP $N+2, 3, N+4$. We see that $5, 6 \in A$ since there is no way to mod-cover them. Therefore, $3, 4, 7$ are mod-covered by A. This leaves us with no way to mod-cover 1, so $1 \in A$. We see that $9, 10, 11, 12$ are covered and 13 cannot be covered by A. Therefore, $13 \in A$ and $S(A) = S(0, 1, 5, 6, 13)$.

Observe that $2N-1$ is covered by $5, N+2$ and $2N-2$ is covered by $6, N+2$ and $2N-3$. However, neither $N+2$ nor $N+4$ may be used to cover $2N-3$. Therefore, $2N-3$ is necessarily covered by $1, N-1$ which implies $N-1 \in A$. However, we again deduce that $N-1, N+2, N+4$ cannot cover $2N-4$. The only way to cover $2N-4$ requires $N-2 \in A$. This is a contradiction since including $N-2$ in A would introduce the mod-AP $N-2, 0, N+2$. □

Lemma 3.14. *There does not exist a modular set A of modulus $2N$ and $\lambda(A) = 9$ with $N+1, N+2, N+3 \notin A$.*

Proof. Let A be a modular set modulo $2N$ with $\lambda(A) = 9$ and $N+1, N+2, N+3 \notin A$. We see that 8 is mod-covered by A and $4 \notin A$. The element 2 is necessarily in A in order to mod-cover 4. The element $7 \in A$ is needed to mod-cover 1. We break our proof into the cases where either (Case I) $3 \in A$ or (Case II) $5 \in A$.

Case I: Since $3 \in A$, we see that 5 is mod-covered by A and $9 \in A$ since it cannot be mod-covered by A. We deduce that $S(A) = S(0, 2, 3, 7, 9) = \{0, 2, 3, 7, 9, 10, 19, \dots\}$. Now, $2N-1$ is covered by $9, N+4$ and $2N-2$ is covered by $10, N+4$. However, $2N-3$ cannot be covered using $N+4$ and can only be covered by $1, N-1$. This is a contradiction since $1 \notin A$.

Case II: Since $5 \in A$, we see that $3, 9, 10$ are mod-covered by A and 11 cannot be mod-covered by A. Therefore, $11 \in A$ and $S(A) = S(0, 2, 5, 7, 11)$. Since $9 \notin A$, we cannot use $N+4$ to cover $2N-1$. Hence $2N-1$ cannot be covered, a contradiction.

Therefore, a modular set A of modulus $2N$ with $\lambda(A) = 9$ and $N+1, N+2, N+3 \notin A$ cannot exist. □

3.4. **Character $\lambda = 11$.** Throughout the remainder of the paper, we will frequently write "covered" or "mod-covered" to mean "covered by A" or "mod-covered by A."

Proposition 3.15. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 11$.*

We break the proof of Proposition 3.15 into Lemmas 3.16, 3.17, 3.18, 3.19, 3.20, and 3.21. When $\lambda(A) = 11$, observe that $\max(A) = N+5$.

Lemma 3.16. *Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ with $N+2 \in A$. Then $N+4 \notin A$.*

Proof. Assume $N+4 \in A$. Observe that $4, 8, 10$ are mod-covered and $2, 3, 5, N+3 \notin A$. Contradiction. There is no way to mod-cover 5. □

Lemma 3.17. *Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ with $N+2 \notin A$. Then $N+4 \notin A$.*

Proof. Assume $N+4 \in A$. Observe that $8, 10$ are mod-covered and $4, 5, N+3 \notin A$. We see that $3 \in A$ is needed to mod-cover 5 and thus $6, 7$ are also mod-covered. Since 2 is required to mod-cover 4, we have $2 \in A$ and $1 \notin A$. We need $9 \in A$ to mod-cover 1 and $11 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0, 2, 3, 9, 11)$, a modular Stanley sequence with character 20. This is a contradiction with $\lambda(A) = 11$. □

Observe that Lemmas 3.16 and 3.17 imply that a modular set A modulo $2N$ with $\lambda(A) = 11$ cannot contain the element $N+4$.

Lemma 3.18. *Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ with $N+4 \notin A$. Then $N+2 \notin A$.*

Proof. Assume $N+2 \in A$. Observe that $4, 10$ are mod-covered and $2, 5 \notin A$. Every possible mod-cover of 5 includes 1, so $1 \in A$ and therefore $2, 3, 9$ are also mod-covered. We then see that $N+3 \in A$ is required to mod-cover 5. Therefore, $N+1 \notin A$ and 6 is mod-covered. We cannot mod-cover $7, 8, 11$, so they are elements of A. Therefore, $S(A) = S(0, 1, 7, 8, 11)$.

We see that $2N-1, 2N-2, \ldots, 2N-9$ are covered. However, we must include an additional element into A in order to cover $2N-10$. The possible candidates are $N-5, N-4, N-3, N-2, N-1$. However, $N-5, N-3, N-2, N-1$ are not allowed for they would introduce a mod-AP into A. Therefore, $2N-10$ is covered by $2, N-4$. This is a contradiction with $2 \notin A$. ☐

Lemma 3.19. *Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ with $N+2, N+4 \notin A$. Then $N+3 \notin A$.*

Proof. Assume $N+3 \in A$. Observe that $6, 10$ are mod-covered and $3, 4, 5, N+1 \notin A$. We require $1 \in A$ to mod-cover 5, so $1 \in A$ and therefore $2, 9$ are also mod-covered. This is a contradiction since there is no way to mod-cover 4. ☐

Lemma 3.20. *Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ with $N+2, N+3, N+4 \notin A$. Then $N+1 \notin A$.*

Proof. Assume $N+1 \in A$. Observe that $2, 10$ are mod-covered and $1, 3, 5 \notin A$. There is no way to mod-cover $5 \notin A$. Contradiction. ☐

Lemma 3.21. *There does not exist a modular set A of modulus $2N$ and $\lambda(A) = 11$ with $N+1, N+2, N+3, N+4 \notin A$.*

Proof. Let A be a modular set modulo $2N$ with $\lambda(A) = 11$ and $N+1, N+2, N+3, N+4 \notin A$. Observe that 10 is mod-covered and $5 \notin A$. We see that 5 must be covered by (Case I) $1, 3$ or (Case II) $3, 4$. In both cases, $3 \in A$, so 6 and 7 are mod-covered.

Case I: In this case $1 \in A$ which implies, $2, 4, 9$ are mod-covered. We see that $4 \in A$ since it cannot be mod-covered, so 8 is covered. Since 11 also cannot be mod-covered, we have $11 \in A$ and $S(A) = S(0, 1, 3, 4, 11)$.

Case II: In this case $4 \in A$ which implies $5, 6, 8$ are mod-covered and $2 \notin A$. We see that 1 is required to cover 2 and in turn $2, 7, 9$ are mod-covered. Since 11 cannot be mod-covered, we have $11 \in A$ and $S(A) = S(0, 1, 3, 4, 11)$.

In both these cases, we have $S(A) = S(0, 1, 3, 4, 11)$. Now, we examine how A covers $2N-1, 2N-2, \ldots$. The elements $2N-1, 2N-2$ are covered by $11, N+5$ and $12, N+5$. However, $2N-3$ requires $N-1 \in A$. Using similar reasoning, one observes that $2N-4, 2N-5, 2N-6$ are covered. However, covering $2N-7$ requires $N-2$ or $N-3$. We cannot include $N-3$ in A otherwise it would contain the mod-AP $N-3, 1, N+5$. Therefore, $N-2 \in A$ and $2N-7$ is covered by $3, N-2$. We see that $2N-8$ is covered but $2N-9$ requires $N-4 \in A$. Even after including $N-4 \in A$, we need $N-5$ to cover $2N-10$. This is a contradiction since the set A would then include the mod-AP $N-5, 0, N+5$.

Therefore, there does not exist a modular set of modulus $2N$ with $\lambda(A) = 11$ and $N+1, N+2, N+3, N+4 \notin A$. ☐

3.5. Character $\lambda = 15$.

Proposition 3.22. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$.*

We break the proof of Proposition 3.22 into Lemmas 3.23 through 3.40. When $\lambda(A) = 15$, observe that $\max(A) = N + 7$.

Lemma 3.23. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 1 \in A$. Then $N + 3 \notin A$.*

Proof. Suppose $N+3 \in A$. Then $2, 6, 14$ are mod-covered and $1, 3, 7, N-7, N-4, N-3, N-1, N+2, N+4, N+5 \notin A$. Necessarily 7 is mod-covered by $5, N+6 \in A$; therefore, $1, 9, 10, 12$ are mod-covered and $N-4, N-6 \notin A$. We need $11 \in A$ to mod-cover 3 which implies $8 \notin A$. Furthermore, $13 \in A$ since it cannot be mod-covered. We see that $4 \in A$ in order to cover 8. We then see that $S(A) = S(0, 4, 5, 11, 13, 16)$. One computes that $2N - 1, \dots, 2N - 11$ are covered. However, $2N - 12$ and therefore must be covered by $x < y$ with $x, y \in A$. However, $y = N - 2$ necessarily which implies $x = 8$, a contradiction with $8 \notin A$. □

Lemma 3.24. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 1 \in A$ and $N + 3 \notin A$. Then $N + 5 \notin A$.*

Proof. Assume $N + 5 \in A$. Then $N + 4, N + 6 \notin A$ and $2, 10, 14$ are mod-covered and $1, 3, 4, 5, 6, 7 \notin A$. This is a contradiction since it is impossible to mod-cover 7. □

Lemma 3.25. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 1 \in A$ and $N + 3, N + 5 \notin A$. Then $N + 6 \notin A$.*

Proof. Assume $N + 6 \in A$. Then $2, 12, 14$ are mod-covered and $1, 4, 6, 7, N - 1, N + 4 \notin A$. We need $5 \in A$ in order to mod-cover 7 which implies $7, 9, 10$ are mod-covered. We see that $8 \in A$ since it cannot be mod-covered and therefore $4, 6, 11$ are mod-covered. Similarly, 3 cannot be mod-covered, so $3 \in A$ and thus 13 is mod-covered. Lastly, $N + 2 \in A$ necessarily to mod-cover 1.

Observe that $S(A) = S(0, 3, 5, 8, 15) = \{0, 3, 5, 8, 15, 17, 18, 20, \dots\}$. However, there is no way to cover $2N - 2 \notin A$. Contradiction. □

Lemma 3.26. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 1 \in A$ and $N + 3, N + 5, N + 6 \notin A$.*

Proof. Assume $N + 2 \in A$. Then $2, 4, 14$ are mod-covered and $1, 7, N + 4 \notin A$. Observe that $3, 5 \in A$ necessarily to cover 7. Therefore, $1, 6, 7, 9, 10, 11$ are mod-covered. We see that $8 \in A$ since it cannot be mod-covered. Therefore, 13 is mod-covered. Lastly, $12 \in A$ since it cannot be covered.

Therefore, $S(A) = S(0, 3, 5, 8, 12, 15)$. We know that $N - 1 \notin A$ otherwise this would introduce mod-AP. However, this leaves us with no way to cover $2N - 2$. Contradiction. □

Lemma 3.27. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ and $N+1 \in A$ and $N + 2, N + 3, N + 5, N + 6 \notin A$.*

Proof. Suppose A is such a modular set. We see that $2, 14$ are mod-covered and $1, 7 \notin A$. Furthermore, $5 \in A$ is needed to cover 7 and $13 \in A$ is needed to mod-cover 1. Therefore, $9, 10$ are covered. In order to mod-cover 7, we require either (Case I) $3 \in A$ or (Case II) $6 \in A$.

Case I: Assume $3 \in A$, then $6, 7, 11$ are mod-covered and $4 \notin A$. This is a contradiction since there is no way to mod-cover 4.

Case II: Assume $6 \in A$, then $7, 8, 12$ are mod-covered and $3, 4 \notin A$. This is a contradiction since there is no way to mod-cover 4.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N + 1 \in A$ and $N + 2, N + 3, N + 5, N + 6 \notin A$. $\square$

Observe that Lemmas 3.23, 3.24, 3.25, 3.26, and 3.27 imply that $N + 1 \notin A$ for a modular set A modulo $2N$ with character $\lambda(A) = 15$.

Lemma 3.28. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 5 \in A$ and $N + 1 \notin A$. Then $N + 4 \notin A$.*

Proof. Suppose $N + 4 \in A$. Then $8, 10, 14$ are mod-covered and $4, 5, 6, 7, N + 3, N + 6 \notin A$. In order to mod-cover 7, we require either (Case I) $1 \in A$ or (Case II) $3 \in A$.

Case I: Assume $1 \in A$, then $2, 7, 9, 13$ are mod-covered. We see $N + 2 \in A$ necessarily to mod-cover 4 and thus 3 is mod-covered. This is a contradiction since there is no way to mod-cover 5.

Case II: Assume $3 \in A$, then $5, 6, 7, 11$ are mod-covered and $N - 1, N + 2 \notin A$. We require $2 \in A$ to cover 4 Therefore, $1 \notin A$ and 12 is mod-covered. We see $13 \in A$ since it cannot be mod-covered which implies 1 is mod-covered. Lastly $9 \in A$ since it cannot be mod-covered.

Therefore, $S(A) = S(0, 2, 3, 9, 13, 19)$. We see that $N - 1$ is needed to cover $2N - 2$. This is a contradiction with $N - 1 \notin A$. $\square$

Lemma 3.29. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 5 \in A$ and $N + 1, N + 4 \notin A$. Then $N + 2 \notin A$.*

Proof. Suppose $N + 2 \in A$. Then $4, 10, 14$ are mod-covered and $2, 5, 6, 7, N + 3, N + 6 \notin A$. We need $3 \in A$ to mod-cover 7 and therefore $1, 6, 11$ are also mod-covered. We deduce $9 \in A$ to mod-cover 5 and then deduce $8 \in A$ since it cannot be mod-covered. Hence, $2, 13$ are mod-covered. Lastly, $12 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0, 3, 8, 9, 12, 17)$. However, $2N - 1$ cannot be covered. Contradiction. $\square$

Lemma 3.30. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N + 5 \in A$ and $N + 1, N + 2, N + 4 \notin A$.*

Proof. Suppose A is such a modular set. Observe that $10, 14$ are mod-covered and $5, 6, 7, N + 3, N + 6 \notin A$. In order to mod-cover 7, we require either (Case I) $1, 4 \in A$ or (Case II) $3 \in A$.

Case I: Assume $1, 4 \in A$. Therefore, $2, 6, 7, 8, 9, 13$ are mod-covered. We require $3 \in A$ to mod-cover 5. Therefore, 11 is mod-covered and $N - 1 \notin A$. We have $12 \in A$ since it cannot be mod-covered. We deduce that $S(A) = S(0, 1, 3, 4, 12, 15)$. This is a contradiction since one cannot cover $2N - 3$.

Case II: Assume $3 \in A$. Observe that $6, 7, 11$ are mod-covered and $N - 1 \notin A$. We break this case into the following four subcases: (Case II.1) $2, 9 \in A$, (Case II.2) $9 \in A$ and $2 \notin A$, (Case II.3) $9 \notin A$ and $1 \in A$, and (Case II.4) $1, 9 \notin A$.

Case II.1: In this case $2, 9 \in A$. We see that $1, 4, 5, 8, 12$ are mod-covered and $13 \in A$ since it cannot be mod-covered. We deduce that $S(A) = S(0, 2, 3, 9, 13, 19)$. This is a contradiction since there is no way to cover $2N - 1$.

Case II.2: In this case $9 \in A$ and $2 \notin A$. We see that $1, 5$ are mod-covered. We see that $12 \in A$ in order to mod-cover 2. We deduce $4 \in A$ since it cannot be mod-covered which implies 8 is mod-covered. Lastly, $13 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0, 3, 4, 9, 12, 13, 16)$ and thus $S(A)$ is an independent Stanley sequence with character $\lambda(A) = 24$. This is a contradiction with $\lambda(A) = 15$.

Case II.3: In this case $1 \in A, 9 \notin A$ and thus $2, 5, 9, 13$ are mod-covered. We see that $4 \in A$ since it cannot be mod-covered and thus 8 is mod-covered. Lastly, we include $12 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0, 1, 3, 4, 12, 15)$. This is a contradiction since there is no way to cover $2N - 3$.

Case II.4: In this case $3 \in A$ and $1, 9 \notin A$. We see that $6, 7, 11$ are mod-covered. We require $4 \in A$ to cover 5. Therefore, $5, 8$ are covered and $2 \notin A$. This is a contradiction since there is no way to mod-cover 9.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N + 5 \in A$ and $N + 1, N + 2, N + 4 \notin A$. $\square$

Observe that Lemmas 3.28, 3.29, and 3.30, along with previous results, imply that $N+5 \notin A$ for a modular set A modulo $2N$ with character $\lambda(A) = 15$.

Lemma 3.31. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 3 \in A$ and $N + 1, N + 5 \notin A$. Then $N + 4 \notin A$.*

Proof. Suppose $N + 4 \in A$. Then $6, 8, 14$ are mod-covered and $3, 4, 5, 7 \notin A$. We need $1 \in A$ to mod-cover 7 and therefore $2, 5, 7, 13$ are mod-covered. We see that $9, 10$ are in A since they cannot be mod-covered and thus $4, 11$ are mod-covered. We need $N + 6 \in A$ to mod-cover 3 which implies 12 is also mod-covered. Thus $S(A) = S(0, 1, 9, 10, 15)$ and $S(A)$ is an independent Stanley sequence with character $\lambda(A) = 24$. This is a contradiction with $\lambda(A) = 15$. $\square$

Lemma 3.32. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N + 3 \in A$ and $N + 1, N + 4, N + 5 \notin A$. Then $N + 2 \notin A$.*

Proof. Suppose $N + 2 \in A$. Then $4, 6, 14$ are mod-covered and $2, 3, 5, 7 \notin A$. This is a contradiction since there is no way to mod-cover 7. $\square$

Lemma 3.33. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N + 3 \in A$ and $N + 1, N + 2, N + 4, N + 5 \notin A$.*

Proof. We see that $6, 14$ are mod-covered and $3, 5, 7, N - 1 \notin A$. We divide the argument into the cases where either (Case I) $11 \in A$ or (Case II) $11 \notin A$.

Case I: In this case, 3 is mod-covered. We see that $1, 4 \in A$ in order to mod-cover 7. Therefore, $2, 5, 6, 7, 8, 10, 13$ are mod-covered and $N + 6 \notin A$. We see that $9, 12 \in A$ since they cannot be mod-covered. Therefore, $S(A) = S(0, 1, 4, 11, 12, 16)$. This is a contradiction since we cannot cover $2N - 1$.

Case II: We need $9, N + 6 \in A$ to mod-cover 3 and therefore $5, 12$ are also mod-covered. We require $1, 4 \in A$ in order to mod-cover 7. Therefore, $2, 8, 10, 11, 13$ are also mod-covered and $N-5, N-4 \notin A$. Hence, $S(A) = S(0,1,4,9,15)$. This is a contradiction since there is no way to cover $2N-11$.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N+3 \in A$ and $N+1, N+2, N+4, N+5 \notin A$. $\square$

Observe that Lemmas 3.31, 3.32, and 3.33 imply that $N+3 \notin A$ for a modular set A modulo $2N$ with character $\lambda(A) = 15$.

Lemma 3.34. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N+2 \in A$ and $N+1, N+3, N+5 \notin A$. Then $N+4 \notin A$.*

Proof. Suppose $N+4 \in A$. Then $4, 8, 14, N+6$ are mod-covered and $2, 3, 7, N-2, N-3 \notin A$. We break our proof into the cases where either (Case I) $1 \in A$ or (Case II) $1 \notin A$.

Case I: Suppose $1 \in A$. Then $2, 3, 7, 13$ are mod-covered. We then have two further subcases: (Case I.1) $5 \in A$ and (Case I.2) $5 \notin A$.

Case I.1: If $5 \in A$, then $9, 10$ are mod-covered. We see that $6 \in A$ since it cannot be mod-covered and thus $11, 12$ are mod-covered. Hence, $S(A) = S(0,1,5,6,15)$. We see that $N-1 \in A$ is necessary to mod-cover $2N-3$. This is a contradiction since there is no way to cover $2N-6$.

Case I.2: If $5 \notin A$, then $9 \in A$ is needed to mod-cover 5. We see that $6 \in A$ since it cannot be mod-covered which implies $11, 12$ are covered. Lastly, $10 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0,1,6,9,10,15)$, an independent Stanley sequence with character $\lambda(A) = 24$. This is a contradiction with $\lambda(A) = 15$.

Case II: If $1 \notin A$, then one requires $5, 6 \in A$ to mod-cover 7 and therefore $2, 3, 7, 9, 10, 12$ are mod-covered. We require $13 \in A$ to mod-cover 1. Lastly, $11 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0,5,6,11,13,18)$. This is a contradiction since there is no way to cover $2N-6$. $\square$

Lemma 3.35. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N+2 \in A$ and $N+1, N+3, N+4, N+5 \notin A$. Then $N+6 \notin A$.*

Proof. Suppose $N+6 \in A$, then $4, 12, 14$ are mod-covered and $2, 6, 7, N-3, N-2 \notin A$. We see that $5 \in A$ in order to cover 7 and therefore $7, 9, 10$ are mod-covered. We conclude that $8 \in A$ since it cannot be mod-covered which implies $6, 11$ are mod-covered. We need $1 \in A$ to cover 2. Hence, $2, 3, 13$ are mod-covered and $N-4, N-5 \notin A$. Therefore, $S(A) = S(0,1,5,8,17)$.

We need $N-1 \in A$ to cover $2N-2$. This is a contradiction since there is no way to cover $2N-11$. $\square$

Lemma 3.36. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N+2 \in A$ and $N+1, N+3, N+4, N+5, N+6 \notin A$.*

Proof. Suppose A is such a modular set. We see that $4, 14$ are mod-covered and $2, 7, N-3, N-2 \notin A$. We need $5 \in A$ to mod-cover 7 and thus $9, 10 \notin A$. We now break the argument up into the cases where either (Case I) $1 \in A$ or (Case II) $1 \notin A$.

Case I: Suppose $1 \in A$. Then $2, 3, 13$ are mod-covered and $N-5 \notin A$. We see that $6 \in A$ to cover 7 and thus $8, 11, 12$ are also mod-covered. Therefore, $S(A) = S(0,1,5,6,15)$. We require $N-1 \in A$ in order to cover $2N-3$. This is a contradiction since there is no way to cover $2N-6$.

Case II: Suppose $1 \notin A$. We need $12 \in A$ to mod-cover 2 and thus $6 \notin A$. We need $3 \in A$ to mod-cover 7 which implies $1, 6, 11$ are mod-covered and $N-1 \notin A$. We see that $8 \in A$ since it cannot be mod-covered and therefore 13 is covered. Therefore, $S(A) = S(0, 3, 5, 8, 12, 15)$. This is a contradiction since one cannot cover $2N-2$.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N+2 \in A$ and $N+1, N+3, N+4, N+5, N+6 \notin A$. ☐

Observe that Lemmas 3.35 and 3.36, along with previous results, imply that $N+2 \notin A$ for a modular set A modulo $2N$ with character $\lambda(A) = 15$.

Lemma 3.37. *Let A be a modular set modulo $2N$ with $\lambda(A) = 15$ with $N+6 \in A$ and $N+1, N+2, N+3, N+5 \notin A$. Then $N+4 \notin A$.*

Proof. Suppose $N+4 \in A$. Then $8, 12, 14$ are mod-covered and $4, 5, 6, 7 \notin A$. We need $1 \in A$ to mod-cover 7 and thus $2, 7, 11, 13$ are mod-covered and $N-2 \notin A$. Therefore we need $3 \in A$ to mod-cover 6 which implies $5, 6, 9$ are mod-covered and $N-1 \notin A$. Lastly, we need $10 \in A$ to mod-cover 4. Thus, $S(A) = S(0, 1, 3, 10, 15)$. This is a contradiction since we cannot cover $2N-5$. ☐

Lemma 3.38. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N+6 \in A$ and $N+1, N+2, N+3, N+4, N+5 \notin A$.*

Proof. Suppose that such a modular set A exists. Observe that $12, 14$ are mod-covered and $6, 7, N-7, N-6 \notin A$. We break our proof up into cases where either (Case I) $5 \in A$ or (Case II) $5 \notin A$.

Case I: Since $5 \in A$ then $7, 9, 10$ are mod-covered. We then break this case up into the subcases where (Case I.1) $4 \in A$, (Case I.2) $3 \in A$, or (Case I.3) $3, 4 \notin A$.

Case I.1: Since $4 \in A$ then $6, 8$ are mod-covered and $2, 3 \notin A$. We need $11 \in A$ to mod-cover 3 which implies 1 is mod-covered. This is a contradiction since there is no way to mod-cover 2.

Case I.2: Since $3 \in A$ then $6, 11$ are mod-covered and $1, 4, N-1 \notin A$. We need $13 \in A$ to mod-cover 1 which then implies that $8 \notin A$. We require $2 \in A$ to mod-cover 4 and 8 which then implies $N-2 \notin A$. Therefore, $S(A) = S(0, 2, 3, 5, 13, 15)$. This is a contradiction since there is no way to cover $2N-5$.

Case I.3: Since $3, 4 \notin A$, we require $8 \in A$ to mod-cover 6. Therefore, $4, 11$ are also mod-covered and $2 \notin A$. This is a contradiction because there is no way to mod-cover 3.

Case II: Since $5 \notin A$, we require $1, 4 \in A$ to mod-cover 7. Therefore, $2, 8, 10, 11, 13$ are also mod-covered and $N-5, N-4 \notin A$. One needs $3 \in A$ to mod-cover 6 which implies that $5, 9$ are also mod-covered and $N-1 \notin A$. Therefore, $S(A) = S(0, 1, 3, 4, 15)$. We need $N-2 \in A$ to cover $2N-8$ and $N-3 \in A$ to cover $2N-9$. This is a contradiction since there is no way to cover $2N-14$.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N+6 \in A$ and $N+1, N+2, N+3, N+4, N+5 \notin A$. ☐

Observe that Lemmas 3.37 and 3.38, along with previous results, imply that $N+6 \notin A$ for a modular set A modulo $2N$ with character $\lambda(A) = 15$.

Lemma 3.39. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N+4 \in A$ and $N+1, N+2, N+3, N+5, N+6 \notin A$.*

Proof. Suppose such a set A exists. Observe that $8, 14$ are mod-covered and $4, 7 \notin A$. We break our proof up into the cases where either (Case I) $1 \in A$ or (Case II) $1 \notin A$.

Case I: Since $1 \in A$, we see that $2, 7, 13$ are mod-covered. We need $10 \in A$ to mod-cover 4 which implies that $5 \notin A$. We see $9 \in A$ since it cannot be mod-covered which implies $5, 11$ are mod-covered. We have $3 \in A$ since it cannot be mod-covered which implies 6 is mod-covered and $N-1 \notin A$. Lastly, $12 \in A$ since it cannot be mod-covered. Therefore, $S(A) = S(0, 1, 3, 9, 10, 12, 16)$. This is a contradiction since one cannot cover $2N-3$.

Case II: Since $1 \notin A$, we need $13 \in A$ to mod-cover 1. We need $5 \in A$ to mod-cover 7 which implies $3, 9, 10$ are mod-covered. Therefore we need $6 \in A$ to mod-cover 7 which implies $2, 12$ are also mod-covered. This is a contradiction since one cannot mod-cover 4.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N+4 \in A$ and $N+1, N+2, N+3, N+5, N+6 \notin A$. $\square$

Lemma 3.40. *There does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ with $N+1, N+2, N+3, N+4, N+5, N+6 \notin A$.*

Proof. Suppose such a set A exists. Then 14 is mod-covered and $7 \notin A$. We break our argument into the case where either (Case I) $5 \notin A$ or (Case II) $5 \in A$.

Case I: If $5 \notin A$ then $1, 4 \in A$ in order to cover 7. Therefore, $2, 7, 8, 10, 13$ are mod-covered by A. We now break this case up into the subcases where (Case I.1) $3 \in A$ and (Case I.2) $3 \notin A$.

Case I.1: If $3 \in A$, then $5, 6, 11$ are mod-covered by A. We see that $9, 12 \in A$ since they cannot be mod-covered. Therefore, $S(A) = S(0, 1, 4, 6, 9, 12, 16)$. This is a contradiction since there is no way to cover $2N-1$.

Case I.2: If $3 \notin A$ then we need $11 \in A$ to mod-cover 3 which implies $6 \notin A$. This is a contradiction because there is no way to mod-cover 6.

Case II: Suppose $5 \in A$. Then $9, 10$ are mod-covered by A. We break this case up into the subcases where (Case II.1) $3 \in A$, (Case II.2) $6 \in A$, $3 \notin A$, and (Case II.3) $3, 6 \notin A$.

Case II.1: Suppose $3 \in A$. Then $6, 7, 11$ are mod-covered and $1, 4, N-1 \notin A$. We need $13 \in A$ to mod-cover 1. We see that $2 \in A$ in order to mod-cover 4 and $8, 12$ are mod-covered as well. Therefore, $S(A) = S(0, 2, 3, 5, 13, 15)$. This is a contradiction since there is no way to cover $2N-3$.

Case II.2: Suppose $6 \in A$ and $3 \notin A$. Then $7, 8, 12$ are mod-covered and $4 \notin A$. We see that $2 \in A$ in order to cover 4 and therefore $1 \notin A$. Thus, $11 \in A$ in order to mod-cover 3 and $13 \in A$ in order to mod-cover 1. Hence, $S(A) = S(0, 3, 5, 6, 11, 13, 18)$. This is a contradiction since there is no way to mod-cover $2N-1$.

Case II.3: Suppose $3, 6 \notin A$. Therefore, $1, 4 \in A$ in order to cover 7. Thus, $2, 6, 7, 8, 13$ are mod-covered and $N-5 \notin A$. We see $11 \in A$ in order to mod-cover 3 and $12 \in A$ since it cannot be mod-covered. Thus, $S(A) = S(0, 1, 4, 5, 11, 12, 15)$. We need $N-1 \in A$ to cover

$2N-3$ and $N-2 \in A$ to cover $2N-4$. Therefore, $N-3 \notin A$. This is a contradiction because there is no way to cover $2N-10$.

Therefore, there does not exist a modular set A modulo $2N$ with $\lambda(A) = 15$ such that $N+1, N+2, N+3, N+4, N+5, N+6 \notin A$. □

4. Future Directions

Though Theorem 1.10 shows that $\lambda(A) \notin \{1, 3, 5, 9, 11, 15\}$ for all independent Stanley sequences $S(A)$, it does not show that every character value $\lambda \in \mathbb{N}_0 \backslash \{1, 3, 5, 9, 11, 15\}$ is achieved by an independent Stanley sequence. In order to prove Conjecture 1.9, one still needs to show that an independent Stanley sequences with character λ exists for every $\lambda \in \mathbb{N}_0 \backslash \{1, 3, 5, 9, 11, 15\}$. Sawhney [5] has recently shown a large subset of even numbers are characters of independent Stanley sequences. However, the case of odd character is still completely open.

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