

# On the Galois group over $\mathbb{Q}$ of a truncated binomial expansion

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**RÉSUMÉ:** Pour tout nombre entier positif  $n$ , les extensions binomiales tronquées de  $(1 + x)^n$  constituées de tous les termes de degré  $\leq r$  où  $1 \leq r \leq n - 2$  semblent toujours être irréductibles. Pour  $r$  fixe et  $n$  suffisamment grand, ce résultat est connu. Nous montrons ici que, pour un nombre entier positif fixe  $r \neq 6$  et  $n$  suffisamment grand, le groupe Galois d'un tel polynôme sur les nombres rationnels est le groupe symétrique  $S_r$ . Pour  $r = 6$ , nous montrons que le nombre de  $n \leq N$  exceptionnels pour lesquels le groupe Galois de ce polynôme n'est pas  $S_r$  est au plus  $O(\log N)$ .

**ABSTRACT:** For positive integers  $n$ , the truncated binomial expansions of  $(1 + x)^n$  which consist of all the terms of degree  $\leq r$  where  $1 \leq r \leq n - 2$  appear always to be irreducible. For fixed  $r$  and  $n$  sufficiently large, this is known to be the case. We show here that for a fixed positive integer  $r \neq 6$  and  $n$  sufficiently large, the Galois group of such a polynomial over the rationals is the symmetric group  $S_r$ . For  $r = 6$ , we show the number of exceptional  $n \leq N$  for which the Galois group of this polynomial is not  $S_r$  is at most  $O(\log N)$ .

## 1 Introduction

For  $t$  and  $r$  non-negative integers, we define

$$p_{r,t}(x) = \sum_{j=0}^r \binom{t+j}{t} x^j = \sum_{j=0}^r \binom{t+j}{j} x^j.$$

This polynomial arises from a normalization of the  $t^{\text{th}}$  derivative of  $1 + x + \cdots + x^{t+r}$ . The polynomial is connected to a factor of the Shabat polynomials of a family of *dessins d'enfant* which are trees and have passport size one (cf. [1, Example 3.3]). The polynomial  $p_{r,t}(x)$  was conjectured

to be irreducible, and the irreducibility was studied in [2]. In particular, we find there that if  $r$  is fixed, then  $p_{r,t}(x)$  is irreducible for  $t$  sufficiently large. A generalization of this irreducibility result can be found in [11], where this polynomial was considered in a different form. There, the irreducibility of the polynomial

$$q_{r,n}(x) = \sum_{j=0}^r \binom{n}{j} x^j,$$

which is a truncated binomial expansion of  $(x+1)^n$ , was investigated. As noted there, this truncated binomial expansion came up in investigations of the Schubert calculus in Grassmannians [20]. Other results concerning these polynomials can be found in [9, 10, 14].

There are some identities involving  $p_{r,t}(x)$  and  $q_{r,n}(x)$  which helped establish the results found in [2] and [11]. If we define

$$\tilde{p}_{r,t}(x) = x^r p_{r,t}(1/x) = \sum_{j=0}^r \binom{t+j}{j} x^{r-j},$$

then according to [2] we have

$$\tilde{p}_{r,t}(x+1) = \sum_{j=0}^r \binom{t+r+1}{j} x^{r-j}.$$

Thus,  $\tilde{p}_{r,t}(x+1) = x^r q_{r,t+r+1}(1/x)$ . We have from [11] that

$$q_{r,n}(x-1) = \sum_{j=0}^r c_j x^j, \quad \text{where } c_j = \binom{n}{j} \binom{n-j-1}{r-j} (-1)^{r-j}.$$

As noted in [11], we can write

$$c_j = \frac{(-1)^{r-j} n(n-1) \cdots (n-j+1)(n-j-1) \cdots (n-r+1)(n-r)}{j!(r-j)!}.$$

These identities are of interest as the irreducibility over  $\mathbb{Q}$  of one of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$  implies the irreducibility of the other three. Furthermore, it is not difficult to see that these polynomials all have the same discriminant (as reversing the coefficients of a polynomial and translating do not affect the discriminant). Also, as the roots for each all generate the same number field, we have that for a fixed  $r$  and  $t$ , the Galois groups over  $\mathbb{Q}$  associated with these polynomials are all the same.

The main goal of this paper is to show that these polynomials give rise to examples of polynomials having Galois group over  $\mathbb{Q}$  the symmetric group.

**Theorem 1.** *Let  $r$  be an integer  $\geq 2$  with  $r \neq 6$ . If  $t$  is a sufficiently large positive integer, then the Galois group associated with any one of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$  over  $\mathbb{Q}$  is the symmetric group  $S_r$ . In the case that  $r = 6$ , there are at most  $O(\log T)$  values of  $t \leq T$  for which the Galois group of any one of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$  over  $\mathbb{Q}$  is not the symmetric group  $S_6$ . In these cases, for sufficiently large  $t$ , the Galois group is  $PGL_2(5)$ , a transitive subgroup of  $S_6$  isomorphic to  $S_5$ .*

Observe that Theorem 1 has as an immediate consequence that for a fixed integer  $r \geq 2$  and  $r \neq 6$ , the Galois group of  $q_{r,n}(x)$  over  $\mathbb{Q}$  is  $S_r$  provided  $n$  is sufficiently large with a similar result for almost all  $n$  in the case that  $r = 6$ . We note that  $q_{6,10}(x)$  has Galois group  $PGL_2(5)$ . The proof of Theorem 1 will give, up to a finite number of exceptions, an explicit description of the set  $\mathcal{N}$  of the  $O(\log T)$  values of  $t \leq T$  where the Galois group  $PGL_2(5)$  might occur. We explain computations that verify directly that for  $10 < n \leq 10^{10}$  and  $n \in \mathcal{N}$ , the Galois group of  $q_{6,n}(x)$  is  $S_6$ . The bound of  $10^{10}$  can easily be extended much further. However, we note that there may still be  $n \in (10, 10^{10}]$  for which  $q_{6,n}(x)$  is reducible so that  $q_{6,n}(x)$  does not have Galois group  $S_6$  since the explicitly given  $\mathcal{N}$  does not take into account that our proof that  $q_{6,n}(x)$  has Galois group  $S_6$  requires  $n$  to be sufficiently large so that, in particular, the results from [2] and [11] imply  $q_{6,n}(x)$  is irreducible. Nevertheless, based on further computations, we conjecture that  $q_{6,n}(x)$  has Galois group  $S_6$  for all  $n \geq 11$ .

## 2 Preliminary Material

We will make use of Newton polygons, which we describe briefly here. Let  $f(x) = \sum_{j=0}^r a_j x^j \in \mathbb{Z}[x]$  with  $a_0 a_r \neq 0$ , and let  $p$  be a prime. For an integer  $m \neq 0$ , we use  $\nu_p(m)$  to denote the exponent in the largest power of  $p$  dividing  $m$ . Let  $S$  be the set of lattice points  $(j, \nu_p(a_{r-j}))$ , for  $0 \leq j \leq r$  with  $a_{r-j} \neq 0$ . The polygonal path along the lower edges of the convex hull of these points from  $(0, \nu_p(a_r))$  to  $(r, \nu_p(a_0))$  is called the Newton polygon of  $f(x)$  with respect to the prime  $p$ . The left-most edge has an endpoint  $(0, \nu_p(a_r))$  and the right-most edge has an endpoint  $(r, \nu_p(a_0))$ . The endpoints of every edge belong to the set  $S$ , and each edge has a distinct slope that increases as we move along the Newton polygon from left to right.

Newton polygons provide information about the factorization of  $f(x)$  over the  $p$ -adic field  $\mathbb{Q}_p$  and, hence, information about the Galois group of  $f(x)$  over  $\mathbb{Q}_p$ . As this Galois group is a subgroup of the Galois group of  $f(x)$  over  $\mathbb{Q}$ , we can use Newton polygons to obtain information about the Galois group of  $f(x)$  over  $\mathbb{Q}$ . Recalling that we are viewing edges of Newton polygons as having distinct slopes, each edge of the Newton polygon of  $f(x)$  with respect to a prime  $p$  corresponds to a factor of  $f(x)$  in  $\mathbb{Q}_p[x]$  that is not necessarily irreducible. More precisely, if an edge has endpoints  $(x_1, y_1)$  and  $(x_2, y_2)$ , then its slope  $a/b = (y_2 - y_1)/(x_2 - x_1)$  with  $\gcd(a, b) = 1$  is such that  $f(x)$  has a factor  $g(x)$  in  $\mathbb{Q}_p[x]$  of degree  $x_2 - x_1$  and with each irreducible factor of  $g(x)$  in  $\mathbb{Q}_p[x]$  of degree a multiple of  $b$ .

We comment here that a theorem of Dedekind (cf. [7]) allows one to obtain information about the Galois group associated with a polynomial  $f(x)$  over  $\mathbb{Q}$  by looking at the polynomials factorization modulo a prime  $p$ . More precisely, suppose  $f(x)$  is an irreducible polynomial in  $\mathbb{Z}[x]$  and  $p$  is a prime which does not divide its discriminant. Suppose further that  $f(x)$  factors modulo  $p$  as a product of  $r$  irreducible polynomials of degrees  $d_1, \dots, d_r$ . Then Dedekind's Theorem asserts that the Galois group of  $f(x)$  over  $\mathbb{Q}$  contains an element that is the product of  $r$  disjoint cycles with cycle lengths  $d_1, \dots, d_r$ .

Our main tool for establishing Theorem 1 is based on combining some of the above ideas with a theorem of C. Jordan [13] and noted in work of R. Coleman [5]. It has been cast in a convenient form by F. Hajir [12], which we summarize as follows.

**Lemma 1.** *Let  $f(x)$  be an irreducible polynomial of degree  $r$ , and suppose  $q$  is a prime in the interval  $(r/2, r - 2)$  such that the Newton polygon with respect to some prime  $p$  has an edge with*

slope  $a/b$  where  $a$  and  $b$  are relatively prime integers and  $q|b$ . Let  $\Delta$  be the discriminant of  $f(x)$ . Then the Galois group of  $f(x)$  over  $\mathbb{Q}$  is the alternating group  $A_r$  if  $\Delta$  is a square and is the symmetric group  $S_r$  if  $\Delta$  is not a square.

We also make use of the following result from [6] and [8] (Theorem 3.3A).

**Lemma 2.** *Let  $f(x)$  be an irreducible polynomial of degree  $r \geq 2$ . If the Galois group of  $f(x)$  over  $\mathbb{Q}$  contains a 2-cycle and a  $q$ -cycle for some prime  $q > r/2$ , then the Galois group is  $S_r$ . Alternatively, if the Galois group of  $f(x)$  over  $\mathbb{Q}$  contains a 3-cycle and a  $q$ -cycle for some prime  $q > r/2$ , then the Galois group is either the alternating group  $A_r$  or the symmetric group  $S_r$ .*

Note that in general if the Galois group of an  $f(x) \in \mathbb{Z}[x]$  over  $\mathbb{Q}$  is contained in an alternating group, then its discriminant is a square. Thus, in the statement of Lemma 2, one can conclude that the Galois group is the symmetric group by showing that the discriminant  $\Delta$  of  $f(x)$  is not a square. In the next section, we give an explicit formula for the discriminant  $\Delta$  of our polynomials in Theorem 1 and show that  $\Delta$  is not a square for fixed  $r$  and for  $t$  sufficiently large. In the last section, for  $r \geq 8$ , we show the existence of primes  $q$  and  $p$  as in Lemma 1. For  $r \leq 7$ , we appeal to Lemma 2 to finish off the proof of Theorem 1.

### 3 The Discriminant

**Lemma 3.** *Let  $t$  and  $r$  be integers with  $t \geq 0$  and  $r \geq 2$ . Let  $\Delta$  be the common discriminant of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$ . Then*

$$\Delta = (-1)^{r(r-1)/2} \frac{(t+1)^{r-1}(t+r+1)^{r-1}(t+2)^{r-2}(t+3)^{r-2} \cdots (t+r)^{r-2}}{(r!)^{r-2}}.$$

*Proof.* We view  $t$  as a variable and work with

$$f_r(x) = q_{r,t+r+1}(x-1) = \sum_{j=0}^r c_j x^j,$$

where

$$c_j = \frac{(-1)^{r-j}(t+r+1) \cdots (t+r-j+2)(t+r-j) \cdots (t+1)}{j!(r-j)!}. \quad (1)$$

To clarify, for  $t$  a non-negative integer, we have

$$c_j = \frac{(-1)^{r-j}(t+r+1)!}{j!(r-j)!t!(t+r-j+1)}.$$

However, from the point of view of (1), we can view  $t$  as a real variable.

We are interested in the discriminant  $\Delta$  of  $f_r(x)$ . Observe that

$$\Delta = \frac{(-1)^{r(r-1)/2}}{c_r} \text{Res}(f_r, f'_r) = \frac{(-1)^{r(r-1)/2} r!}{(t+r+1)(t+r) \cdots (t+3)(t+2)} \text{Res}(f_r, f'_r), \quad (2)$$

where  $\text{Res}(f_r, f'_r)$  is the resultant of  $f_r$  and  $f'_r$  with respect to the variable  $x$ . We express the resultant in terms of the  $(2r - 1) \times (2r - 1)$  Sylvester determinant

$$\text{Res}(f_r, f'_r) = \begin{vmatrix} c_r & c_{r-1} & c_{r-2} & \dots & c_1 & c_0 & 0 & 0 & \dots & 0 \\ 0 & c_r & c_{r-1} & \dots & c_2 & c_1 & c_0 & 0 & \dots & 0 \\ 0 & 0 & c_r & \dots & c_3 & c_2 & c_1 & c_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ rc_r & (r-1)c_{r-1} & (r-2)c_{r-2} & \dots & c_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & rc_r & (r-1)c_{r-1} & \dots & 2c_2 & c_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & rc_r & \dots & 3c_3 & 2c_2 & c_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

Observe that there are  $r - 1$  rows consisting of the coefficients of  $f_r(x)$  and  $r$  rows consisting of the coefficients of  $f'_r(x)$ . For each integer  $j \in [1, r]$ , we see from (1) that  $t + r + 1$  divides  $c_j$ . We deduce that  $t + r + 1$  can be factored out of each element of the first  $r$  columns of  $\text{Res}(f_r, f'_r)$  to show that  $(t + r + 1)^r$  is a factor of  $\text{Res}(f_r, f'_r)$ . For each  $k \in \{1, 2, \dots, r\}$ , we also have from (1) that  $t + r - k + 1$  divides  $c_j$  for each integer  $j \in [1, r]$  with  $j \neq k$ . In particular, each of the  $r - 1$  columns not containing  $kc_k$  have each element divisible by  $t + r - k + 1$ . Thus,  $(t + r - k + 1)^{r-1}$  divides  $\text{Res}(f_r, f'_r)$ . Hence,

$$(t + r + 1)^r (t + 1)^{r-1} (t + 2)^{r-1} \dots (t + r)^{r-1} \quad (3)$$

divides  $\text{Res}(f_r, f'_r)$ . The product in (3) as a polynomial in  $t$  has degree  $(r + 1)(r - 1) + 1 = r^2$ . Hence, from (2), we see that  $\Delta$  is divisible by the polynomial

$$(t + r + 1)^{r-1} (t + 1)^{r-1} (t + 2)^{r-2} (t + 3)^{r-2} \dots (t + r)^{r-2} \quad (4)$$

of degree  $r^2 - r$  in  $t$ .

We turn to making use of

$$p_{r,t}(x) = \sum_{j=0}^r d_j x^j, \quad \text{where } d_j = \sum_{i=0}^r \frac{(t+i)(t+i-1)\dots(t+1)}{i!} x^i.$$

Observe that the discriminant of  $f_r(x)$  and  $p_{r,t}(x)$  are both polynomials in  $t$  that agree at all positive integers and, hence, are identical. We use next that  $\Delta$  is the discriminant of  $p_{r,t}(x)$  to show that  $\Delta$  cannot be divisible by a higher degree polynomial in  $t$  than that given by (4). Taking into account the leading coefficient of  $p_{r,t}(x)$ , we see that

$$\Delta = \frac{(-1)^{r(r-1)/2} r!}{(t+r)(t+r-1)\dots(t+2)(t+1)} \text{Res}(p_{r,t}, p'_{r,t}), \quad (5)$$

where

$$\text{Res}(p_{r,t}, p'_{r,t}) = \begin{vmatrix} d_r & d_{r-1} & d_{r-2} & \dots & d_1 & d_0 & 0 & 0 & \dots & 0 \\ 0 & d_r & d_{r-1} & \dots & d_2 & d_1 & d_0 & 0 & \dots & 0 \\ 0 & 0 & d_r & \dots & d_3 & d_2 & d_1 & d_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ rd_r & (r-1)d_{r-1} & (r-2)d_{r-2} & \dots & d_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & rd_r & (r-1)d_{r-1} & \dots & 2d_2 & d_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & rd_r & \dots & 3d_3 & 2d_2 & d_1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{vmatrix}.$$

Observe that  $d_j$  is a polynomial of degree  $j$  in  $t$  for each  $j \in \{0, 1, \dots, r\}$ . We set  $A = (a_{ij})$  to be the  $(2r-1) \times (2r-1)$  matrix defining  $\text{Res}(p_{r,t}, p'_{r,t})$  above, so

$$a_{ij} = \begin{cases} d_{r+i-j} & \text{if } 1 \leq i \leq r-1 \text{ and } i \leq j \leq i+r \\ (i-j+1)d_{i-j+1} & \text{if } r \leq i \leq 2r-1 \text{ and } i-r+1 \leq j \leq i \\ 0 & \text{otherwise.} \end{cases}$$

We make use of the definition of a determinant to obtain

$$\text{Res}(p_{r,t}, p'_{r,t}) = \det A = \sum_{\sigma \in S_{2r-1}} \left( \text{sgn}(\sigma) \prod_{i=1}^{2r-1} a_{i, \sigma(i)} \right).$$

We show that independent of  $\sigma \in S_{2r-1}$ , the product  $\prod_{i=1}^{2r-1} a_{i, \sigma(i)}$  is a polynomial of degree at most  $r^2$  in  $t$ . In fact, more is true. If each  $a_{i, \sigma(i)} \neq 0$ , then we show that  $\prod_{i=1}^{2r-1} a_{i, \sigma(i)}$  is a polynomial of degree exactly  $r^2$  in  $t$ . Indeed, for such  $\sigma$ , we have

$$\begin{aligned} i \leq \sigma(i) \leq i+r & \quad \text{for } 1 \leq i \leq r-1, \\ i-r+1 \leq \sigma(i) \leq i & \quad \text{for } r \leq i \leq 2r-1, \end{aligned}$$

and

$$\begin{aligned} \deg \left( \prod_{i=1}^{2r-1} a_{i, \sigma(i)} \right) &= \sum_{i=1}^{r-1} \deg(a_{i, \sigma(i)}) + \sum_{i=r}^{2r-1} \deg(a_{i, \sigma(i)}) \\ &= \sum_{i=1}^{r-1} (r+i-\sigma(i)) + \sum_{i=r}^{2r-1} (1+i-\sigma(i)) \\ &= r^2 + \sum_{i=1}^{2r-1} (i-\sigma(i)) = r^2. \end{aligned}$$

We set

$$\rho = \sum_{\sigma \in S_{2r-1}} \left( \text{sgn}(\sigma) \prod_{i=1}^{2r-1} \ell(a_{i, \sigma(i)}) \right),$$

where  $\ell(a_{i,\sigma(i)})$  denotes the leading coefficient of  $a_{i,\sigma(i)}$ . Observe that if  $\rho \neq 0$ , then  $\det A$  is a polynomial of degree  $r^2$  with leading coefficient  $\rho$ . The value of  $\rho$  is the determinant of  $(2r - 1) \times (2r - 1)$  matrix  $(\ell(a_{ij}))$ . Since

$$\ell(a_{ij}) = \begin{cases} 1/(r + i - j)! & \text{for } 1 \leq i \leq r - 1 \text{ and } i \leq j \leq i + r \\ 1/(i - j)! & \text{for } r \leq i \leq 2r - 1 \text{ and } i - r + 1 \leq j \leq i \\ 0 & \text{otherwise,} \end{cases}$$

this determinant is the value of  $\text{Res}(g, g')$ , where  $g(x) = \sum_{j=0}^r x^j/j!$ . This polynomial truncation of  $e^x$  has been studied by R. F. Coleman [5] and I. Schur [21, 22]. In particular,  $g(x)$  corresponds to the generalized Laguerre polynomial  $L_r^{(-r-1)}(x)$ , for which I. Schur [23] gives an explicit formula for the discriminant from which the value of  $\text{Res}(g, g')$  is easily determined (also, see [16], Chapter 9). From these, we see that

$$\text{Res}(g, g') = \frac{1}{(r!)^{r-1}}.$$

We deduce that  $\text{Res}(p_{r,t}, p'_{r,t})$  is a polynomial of degree  $r^2$  in  $t$  with leading coefficient  $1/(r!)^{r-1}$ . From (5), we see that  $\Delta$  is a polynomial of degree  $r^2 - r$  in  $t$  which has leading coefficient  $(-1)^{r(r-1)/2}/(r!)^{r-2}$ . Since (4) divides  $\Delta$ , the lemma follows.  $\square$

**Lemma 4.** *Let  $r$  be an integer  $\geq 2$ . For  $t$  a non-negative integer, let  $\Delta$  be the common discriminant of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$ . Then there is a  $t_0 = t_0(r)$  such that for all  $t \geq t_0$ , the value of  $\Delta$  is not a square.*

*Proof.* Suppose  $r \geq 2$  and  $t \geq 0$  are such that  $\Delta$  is a square. From Lemma 3, we see that  $r \geq 4$  since  $\Delta < 0$  for  $r \in \{2, 3\}$ . We consider even and odd  $r$  separately and only the case that  $\Delta \geq 0$  since  $\Delta < 0$  cannot be a square.

In the case that  $r$  is even, Lemma 3 implies that  $(t+1)(t+r+1)$  is an integer that is a rational square. Hence,  $(t+1)(t+r+1)$  is the square of an integer. Let  $\delta = \gcd(t+1, t+r+1)$ . Then  $\delta$  divides the difference  $(t+r+1) - (t+1) = r$ , so  $\delta \leq r$ . Also  $(t+1)/\delta$  and  $(t+r+1)/\delta$  are relatively prime numbers whose product is a square, so each of them is a square. As  $(t+r+1)/\delta - (t+1)/\delta = r/\delta \leq r$  and the difference of two consecutive squares  $(n+1)^2 - n^2 = 2n+1$  tends to infinity with  $n$ , we deduce that  $(t+1)/\delta$  is bounded. In fact, taking  $n^2 = (t+1)/\delta$ , we see that

$$\begin{aligned} 2\sqrt{\frac{t+1}{r}} + 1 &\leq 2\sqrt{\frac{t+1}{\delta}} + 1 \leq \frac{r}{\delta} \leq r \implies \frac{4(t+1)}{r} \leq (r-1)^2 \\ &\implies t < t+1 \leq \frac{r(r-1)^2}{4}. \end{aligned}$$

Thus, for  $r$  even and  $t \geq r(r-1)^2/4$ , we have that  $\Delta$  is not a square.

In the case that  $r$  is odd, Lemma 3 implies that the largest factor of the product

$$(t+2)(t+3) \cdots (t+r)$$

relatively prime to  $r!$  is a square. As  $(t+r) - (t+2) = r-2$ , we see also that for every prime  $p > r$  dividing the product  $(t+2)(t+3) \cdots (t+r)$ , there is a unique  $j \in \{2, 3, \dots, r\}$  for which

$p|(t+j)$ . Thus, for such a  $p$  and  $j$ , there is a positive integer  $e$  for which  $p^{2e}||t+j$ . As  $r \geq 5$ , we deduce that there are positive integers  $a, b$  and  $c$  each dividing the product of the primes up to  $r$  and satisfying

$$t+2 = au^2, \quad t+3 = bv^2 \quad \text{and} \quad t+4 = cw^2,$$

for some positive integers  $u, v$  and  $w$ . We deduce that

$$b^2v^4 - 1 = (t+3)^2 - 1 = (t+2)(t+4) = ac(uw)^2.$$

As  $a, b$  and  $c$  divide the product of the primes up to  $r$ , there are finitely many equations of the form  $acy^2 = b^2x^4 - 1$  possible for a given  $r$ . Each such equation is an elliptic curve containing finitely many integral points by a theorem of Siegel [15, 25, 26]. Hence, for a fixed  $r$ , the value of  $x = v = \sqrt{(t+3)/b}$  is bounded from above for every possible  $b$ . Thus,  $t_0$  exists in the case of  $r$  odd, completing the proof.  $\square$

## 4 Proof of Theorem 1

As noted in the introduction, from [2] and [11], for  $t$  sufficiently large, the polynomial  $p_{r,t}(x)$  and, hence, the polynomials  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$  are irreducible.

We begin by considering the case that  $r$  is a fixed integer  $\geq 8$ . From Lemma 1 and Lemma 4, it suffices to show that there is a prime  $q$  in the interval  $(r/2, r-2)$  such that the Newton polygon of one of  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x+1)$  and  $q_{r,t+r+1}(x-1)$  with respect to some prime  $p$  has an edge with slope  $a/b$  where  $a$  and  $b$  are relatively prime integers and  $q|b$ .

Since  $r \geq 8$ , one can show using explicit results on the distribution of primes (cf. [19]) that there is a prime  $q \in (r/2, r-2)$ . Alternatively, from [18], one has that there are 3 primes in the interval  $(r/2, r]$  for  $r \geq 17$  so that there must be at least 1 prime in the interval  $(r/2, r-2)$  for  $r \geq 17$ . Then a simple check leads to such a prime for  $r \geq 8$ .

With  $q$  a prime in  $(r/2, r-2)$ , we consider  $t \in \mathbb{Z}^+$  sufficiently large. Note that the numbers  $t+r+1-q$  and  $t+1+q$  are distinct positive integers. Let  $p$  be a prime  $> r$ , and suppose  $p^e || (t+r+1-q)(t+1+q)$  where  $e \in \mathbb{Z}^+$ . Observe that  $p > r$  implies either  $p^e || (t+r+1-q)$  or  $p^e || (t+1+q)$ . We use that in fact  $p$  can divide at most one of  $t+r+1, t+r, \dots, t+1$ . Suppose  $p^e || (t+r+1-q)$ . Then the Newton polygon of  $q_{r,t+r+1}(x-1)$  with respect to  $p$  consists of two edges, one joining  $(0, e)$  to  $(r-q, 0)$  and the other joining  $(r-q, 0)$  to  $(r, e)$ . In the case that  $p^e || (t+1+q)$ , the Newton polygon of  $q_{r,t+r+1}(x-1)$  with respect to  $p$  also consists of two edges, one joining  $(0, e)$  to  $(q, 0)$  and the other joining  $(q, 0)$  to  $(r, e)$ . In either case, we see that the Newton polygon of  $q_{r,t+r+1}(x-1)$  with respect to  $p$  has an edge of slope  $\pm e/q$ . From Lemma 1, we can therefore deduce for sufficiently large  $t$ , the Galois group of  $q_{r,t+r+1}(x-1)$  is  $S_r$  unless  $q|e$ . This is true for each prime  $p > r$  with  $p|(t+r+1-q)(t+1+q)$ .

So suppose then that for every prime  $p > r$  dividing  $(t+r+1-q)(t+1+q)$ , we have  $p^e || (t+r+1-q)$  or  $p^e || (t+1+q)$  for some  $e$  divisible by  $q$ . We deduce that we can write

$$t+1+q = au^q \quad \text{and} \quad t+r+1-q = bv^q,$$

where  $a, b, u$  and  $v$  are positive integers with both  $a$  and  $b$  dividing

$$\mathcal{P} = \prod_{\substack{p \leq r \\ p \text{ prime}}} p^{q-1}.$$



Note that  $q \in (r/2, r-2)$  and  $r \geq 8$ , so that  $q \geq 5$ . For fixed  $a$  and  $b$  dividing  $\mathcal{P}$ , we have  $u$  and  $v$  must be solutions to the Diophantine equation

$$au^q - bv^q = 2q - r > 0.$$

As this is a Thue equation, we deduce that there are finitely many integral solutions in  $u$  and  $v$  (cf. [24]). This is true for each fixed  $a$  and  $b$  dividing  $\mathcal{P}$ . As  $\mathcal{P}$  and  $q$  only depend on  $r$  and  $r$  is fixed, we deduce that there are finitely many possibilities for  $t+1+q = au^q$ . Hence, for sufficiently large  $t$ , we deduce that  $q \nmid e$  for some prime  $p > r$  with  $p^e \parallel (t+r+1-q)$  or  $p^e \parallel (t+1+q)$ . Consequently, in the case that  $r \geq 8$ , we can conclude the Galois group of  $q_{r,t+r+1}(x-1)$  is  $S_r$ , from which the same follows for the polynomials  $p_{r,t}(x)$ ,  $\tilde{p}_{r,t}(x)$  and  $\tilde{p}_{r,t}(x+1)$ .

Now, we consider the case that  $r \leq 7$ . We consider  $t$  sufficiently large so that in particular the polynomials in Theorem 1 are irreducible. In the case that  $r = 2$ , the only possibility then is that the Galois group is  $S_2$ . For  $r = 3$ , we use also that, by Lemma 4, the discriminant of  $p_{3,t}(x)$  is not a square, and this is enough to imply that the Galois group of  $p_{3,t}(x)$  over  $\mathbb{Q}$  is  $S_3$ . For  $r = 4$ , suppose  $p$  is a prime  $> 3$  dividing  $(t+2)(t+4)$ . If  $p^e \parallel (t+2)(t+4)$ , then  $p^e \parallel (t+2)$  or  $p^e \parallel (t+4)$ . In either case, we see that the Newton polygon of  $q_{r,t+r+1}(x-1)$  with respect to  $p$  consists of an edge with slope  $e/3$ . If  $3 \nmid e$ , then the Galois group will have a 3-cycle so that Lemma 2 and Lemma 4 imply that the Galois group is  $S_4$ . Otherwise,  $3 \mid e$  for each prime  $p > 3$  dividing  $(t+2)(t+4)$ . We deduce that  $t+4 = au^3$  and  $t+2 = bv^3$  where  $a$  and  $b$  divide 36. Observe that  $au^3 - bv^3 = 2$ . This is a Thue equation, and as before this equation has no solutions for  $t$  sufficiently large. Thus, since  $t$  is sufficiently large, the Galois group of  $p_{4,t}(x)$  over  $\mathbb{Q}$  is  $S_4$ .

For  $r = 5$  and  $r = 7$ , one can give similar arguments. Specifically, for  $r = 5$  and for a prime  $p > 3$  such that  $p^e \parallel (t+3)(t+4)$ , we deduce that either  $3 \mid e$  or else there is a  $\sigma$  in the Galois group of  $p_{5,t}(x)$  over  $\mathbb{Q}$  which is a 3-cycle or is a product of two disjoint cycles, one a 3-cycle and one a 2-cycle. In the case that  $3 \mid e$  for every such prime  $p > 3$ , we have  $t+4 = au^3$ ,  $t+3 = bv^3$  and  $au^3 - bv^3 = 1$ , where  $a$  and  $b$  divide 36. Since  $t$  is sufficiently large, this does not occur. In the case that  $\sigma$  is a product of a 3-cycle and a 2-cycle, we see that  $\sigma^2$  is a 3-cycle. Thus, regardless of  $\sigma$ , we can apply Lemma 2 and Lemma 4 to deduce that the Galois group of  $p_{5,t}(x)$  over  $\mathbb{Q}$  is  $S_5$ . For  $r = 7$  and for a prime  $p > 3$  such that  $p^e \parallel (t+4)(t+5)$ , one similarly argues that either  $3 \mid e$  for every prime  $p > 3$  and a Thue equation shows that this impossible since  $t$  is sufficiently large or there is a  $\sigma$  in the Galois group of  $p_{7,t}(x)$  over  $\mathbb{Q}$  such that  $\sigma^4$  is a 3-cycle. Also, for  $r = 7$  and for a prime  $p > 7$  such that  $p^e \parallel (t+1)(t+8)$ , one similarly argues that either  $7 \mid e$  for every prime  $p > 7$  and a Thue equation shows that this impossible since  $t$  is sufficiently large or there is a 7-cycle in the Galois group of  $p_{7,t}(x)$  over  $\mathbb{Q}$ . Thus, the Galois group of  $p_{7,t}(x)$  over  $\mathbb{Q}$  contains a 3-cycle and a 7-cycle, and Lemma 2 and Lemma 4 imply this Galois group is  $S_7$ .

We are left with the case that  $r = 6$ . There are 16 transitive subgroups of  $S_6$  (cf. [8]). We can eliminate all but two of these as possibilities for the Galois group  $G$  of  $p_{6,t}(x)$  over  $\mathbb{Q}$  as follows. Using an argument similar to the above, we consider a prime  $p > 5$  such that  $p^e \parallel (t+2)(t+6)$  to show that either  $5 \mid e$  for every prime  $p > 5$  and a Thue equation shows that this impossible since  $t$  is sufficiently large or there is a 5-cycle in  $G$ . Since  $t$  is sufficiently large, we deduce that  $p_{6,t}(x)$  is irreducible over  $\mathbb{Q}$ ,  $p_{6,t}(x)$  has a non-square discriminant in  $\mathbb{Q}$ , and  $G$  contains a 5-cycle. The latter implies that 5 divides  $|G|$ . Of the 16 transitive subgroups of  $S_6$ , only 4 have size divisible by 5, and of those exactly 2 are contained in  $A_5$ . Since the discriminant of  $p_{6,t}(x)$  is not a square, this leaves then just 2 possibilities for  $G$ , one is  $S_6$  and the other is  $PGL_2(5)$ , which is a subgroup of  $S_6$  that is isomorphic to  $S_5$ .

For the purposes of the proof of Theorem 1, we can distinguish between cases where  $G = S_6$  and cases where  $G = PGL_2(5)$  by observing that  $S_6$  has an element which is the product of two disjoint cycles, one a 2-cycle and the other a 4-cycle, whereas  $PGL_2(5)$  has no such element. We consider a prime  $p > 3$  such that  $p^e \parallel (t+3)(t+5)$  for some  $e \in \mathbb{Z}^+$ .

If  $2 \nmid e$  then  $p_{6,t}(x) = g(x)h(x)$  where  $g(x)$  and  $h(x)$  are irreducible polynomials over  $\mathbb{Q}_p$  of degrees 2 and 4 respectively. Let  $F_g$  and  $F_h$  denote the splitting fields of  $g$  and  $h$  over  $\mathbb{Q}_p$  and observe that they are tamely ramified since  $p > 3$ . Using Newton polygons, one deduces that  $F_g$  is totally ramified and that the ramification index of  $F_h$  is divisible by 4. We know that  $F_h$  is tamely ramified and therefore the tame inertia group is cyclic with order divisible by 4 [3, Corollary 1, p. 31]. Since  $S_4$  has no larger cyclic subgroups, we deduce that the ramification index of  $F_h$  is exactly 4 and the tame inertia subgroup is generated by a 4-cycle (the only possible form of an element with order 4 in  $S_4$ ).

Now, let  $K$  be the compositum of  $F_g$  and  $F_h$ . If  $F_h \subsetneq K$ , then  $F_h \cap F_g = \mathbb{Q}_p$ . Therefore, there is an element of the Galois group of  $p_{6,t}(x)$  that permutes the 2 roots of  $g$  and cyclicly permutes the 4 roots of  $h$ . That is, the Galois group of  $p_{6,t}(x)$  contains an element which is the disjoint product of a 4-cycle and a 2-cycle.

If  $K = F_h$ , then  $F_g \subset F_h$ . Let  $\tau$  be a generator of the tame inertia group of  $F_h$ . If  $\tau$  permutes the roots  $g$ , then the Galois group of  $p_{6,t}(x)$  contains an element which is the disjoint product of a 4-cycle and a 2-cycle. If  $\tau$  fixes the roots of  $g$ , then the roots of  $g$  lie in  $K^\tau$ , the fixed field of  $\tau$ . However, since  $\tau$  generates the inertia subgroup of  $K$ , we know that  $K^\tau$  is an unramified extension of  $\mathbb{Q}_p$  [17, Proposition 9.11, p. 173]. Therefore,  $F_g \subset K^\tau$  is unramified, which is a contradiction with our previous deduction that  $F_g$  is a totally ramified quadratic extension of  $\mathbb{Q}_p$ .

Since the Galois group  $G$  has an element that is the product of two disjoint cycles, one a 2-cycle and the other a 4-cycle, we have shown that  $G = S_6$ . In the case that  $2 \mid e$  for every prime  $p > 3$ , we have  $t+5 = au^2$  and  $t+3 = bv^2$  where  $a$  and  $b$  are divisors of 6. In this case we have, for fixed  $a$  and  $b$ , the Diophantine equation

$$au^2 - bv^2 = 2 \tag{6}$$

in the variables  $u$  and  $v$ . Of the 16 possibilities for  $(a, b)$  where  $a$  and  $b$  divide 6, there are 9 for

Pairs $(a, b)$	All Solutions
(1, 2)	$u = 2u'$ where $(1 + \sqrt{2})^{2m-1} = v + \sqrt{2}u'$ for $m \in \mathbb{Z}^+$
(2, 1)	$v = 2v'$ where $(1 + \sqrt{2})^{2m} = u + \sqrt{2}v'$ for $m \in \mathbb{Z}^+$
(2, 3)	$v = 2v'$ where $(5 + 2\sqrt{6})^m = u + \sqrt{6}v'$ for $m \in \mathbb{Z}^+$
(2, 6)	$(2 + \sqrt{3})^m = u + \sqrt{3}v$ for $m \in \mathbb{Z}^+$
(3, 1)	$(1 + \sqrt{3})(2 + \sqrt{3})^{m-1} = v + \sqrt{3}u$ for $m \in \mathbb{Z}^+$
(6, 1)	$(2 + \sqrt{6})(5 + 2\sqrt{6})^{m-1} = v + \sqrt{6}u$ for $m \in \mathbb{Z}^+$

Table 1: Solutions to the Pell Equations

which (6) can be shown to have no solutions modulo either 3 or 4. For  $(a, b) = (2, 2)$ , the equation (6) is equivalent to  $u^2 - v^2 = 1$ . Since consecutive positive squares differ by more than 1 and since  $t+5 = au^2$ , we deduce that there are no solutions for  $t \geq 1$ . The remaining 6 choices of  $(a, b)$

are tabulated in Table 1. Here, the equation (6) corresponds to a Pell equation which has infinitely many solutions in *positive* integers  $u$  and  $v$  given by the right column in the table. These solutions were found using classical methods for solving Pell equations (cf. [4]), and we do not elaborate on the details. In each case, the solutions grow exponentially, and the total number of solutions in pairs  $(u, v)$  with  $u$  and  $v$  each  $\leq X$  is  $O(\log X)$ . As  $t + 5 = au^2$  and  $t + 3 = bv^2$ , we deduce that the number of  $t \leq T$  such that  $G = PGL_2(5)$  is at most  $O(\log T)$ , completing the proof of Theorem 1.

The inclusion of the phrase “at most” in the theorem is to emphasize that we do not know that these exceptional pairs that arose at the end of this proof give rise to cases where  $G = PGL_2(5)$ . In fact, it is likely that  $G = S_6$  for every sufficiently large  $t$  when  $r = 6$ . For  $t \in \{1, 3\}$ , which arise from the two smallest solutions coming from Table 1, one checks that  $G = PGL_2(5)$ . There are 37 other positive integer values of  $t \leq 10^{10}$  coming from Table 1, and one checks that for each of these we have:

- For some prime  $p_1 \leq 149$ , the polynomial  $p_{6,t}(x)$  is an irreducible sextic polynomial modulo  $p_1$ . Hence,  $p_{6,t}(x)$  is irreducible.
- The discriminant  $\Delta$  of  $p_{6,t}(x)$  is not a square. Hence,  $G$  is not contained in  $A_6$ . (Note that by Lemma 3, if  $r = 6$ , then  $\Delta < 0$  for all non-negative integers  $t$ ; thus,  $\Delta$  cannot be a square if  $r = 6$ .)
- For some prime  $p_2 \leq 101$ , the polynomial  $p_{6,t}(x)$  factors as a linear polynomial times an irreducible quintic modulo  $p_2$ . Hence,  $G = PGL_2(5)$  or  $G = S_6$ .
- For some prime  $p_3 \leq 109$  not dividing the discriminant  $\Delta$  of  $p_{6,t}(x)$ , the polynomial  $p_{6,t}(x)$  factors as an irreducible quadratic times an irreducible quartic modulo  $p_3$ . Hence,  $G = S_6$  (using Dedekind’s Theorem discussed in Section 2).

It therefore is plausible that for  $t > 3$  in general, the Galois group of  $p_{6,t}(x)$  is in fact  $S_6$ . Note that since  $\tilde{p}_{r,t}(x + 1) = x^r q_{r,t+r+1}(1/x)$ , the comments about  $q_{r,n}(x)$  after the statement of Theorem 1 follow.

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