

# Monodromy Groups of Dessins d'Enfant on Rational Triangular Billiards Surfaces

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## Abstract

A *dessin d'enfant*, or *dessin*, is a bicolored graph embedded into a Riemann surface, and the monodromy group is an algebraic invariant of the dessin generated by rotations of edges about black and white vertices. A *rational billiards surface* is a two dimensional surface that allows one to view the path of a billiards ball as a continuous path. In this paper, we classify the monodromy groups of dessins associated to rational triangular billiards surfaces.

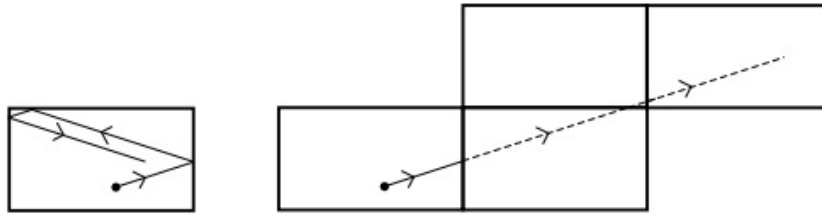
## Introduction

A rational billiards surface is a two dimensional surface that allows one to view the path of a billiards ball as a continuous path instead of a jagged path obtained from numerous bounces off the sides of a billiards table. As one changes the shape of the billiards table, one obtains different billiards surfaces. In [4], the authors studied the Cayley graph associated to billiards surfaces obtained from rational triangular billiards tables. In this project, we propose modifying their approach by classifying the monodromy groups of *dessins d'enfant* drawn on these billiards surfaces. In particular, we prove these groups are semi-direct products of abelian groups.

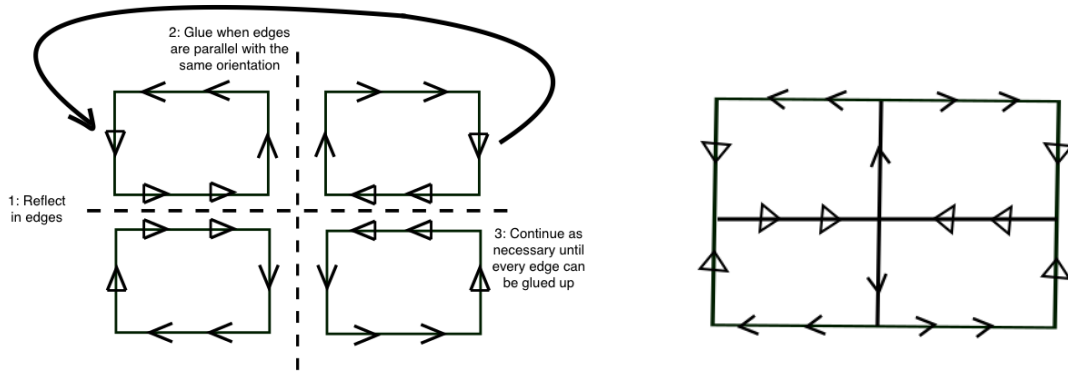
## Background

### Billiard Surfaces

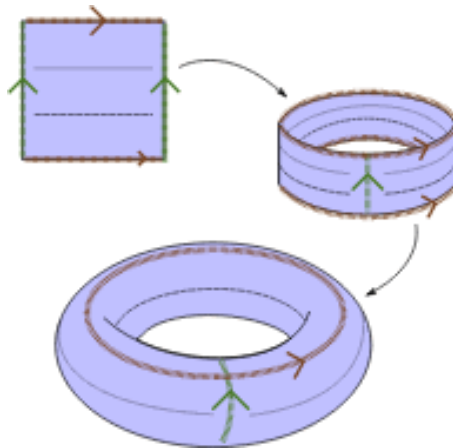
A basic definition of a billiards surface is a Riemann surface constructed from a polygon with angles in radians that are rational multiples of  $\pi$  [3]. A billiards surface is a topological construction that allows one to view the path of a billiards ball as a continuous path on a surface instead of a chaotic path of bounces off the sides of the billiards table. This concept is best illustrated via a picture involving the billiards surface associated with the rectangle. One constructs the billiards surface by beginning with one copy of the rectangle and then reflecting it a certain number of times as pictured below. This is referred to as unfolding the path.



Once one has enough copies of the initial polygon, one identifies edges in the surface that are parallel and have the same orientation. In essence, this amounts to gluing together pairs of sides of our diagram. Here is an example of how edges are identified in the billiards surface for the equilateral right triangle.



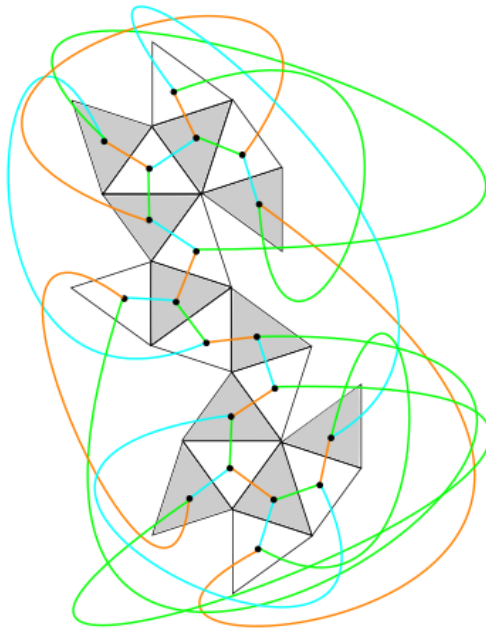
One obtains the following 1-torus after gluing together the opposite sides of the larger rectangle.



In fact, all billiards surfaces are tori with one or more holes [5].

### Graphs on Surfaces

As studied in [4], one may construct a graph on the billiards surface. The Cayley graph of a billiards surface is the graph obtained by placing a vertex at the center of each polygon and edges are drawn between vertices if their corresponding polygons are adjacent. An example of the Cayley graph of the billiards surface corresponding to the  $\frac{3\pi}{10}, \frac{3\pi}{10}, \frac{4\pi}{10}$  triangle is shown below.



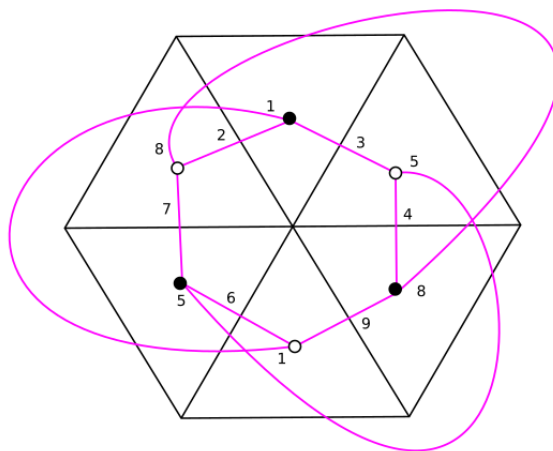
A billiards surface can have arbitrarily large genus. However, the genus of the Cayley graph of a triangular billiards surface is always zero or one [4]!

Although the Cayley graphs are an interesting object to study, they lose some of the geometric information of the original billiards surface as evidenced by the fact that the genus of the Cayley graph of a triangular billiards surface is at most one. In order to create a graph that preserves more of the structure of the billiards surface, we introduce the notion of a dessin d'enfant.

A *dessin d'enfant*, or simply *dessin*, is a connected bicolored (e.g. the vertices are one of two colors) graph equipped with a cyclic ordering of the edges (oriented counterclockwise) around each vertex [2]. In the case of the Cayley graph, one can rearrange the order of the edges coming out of a vertex without changing the graph. However, this is not allowed in the case of dessins d'enfant.

One important algebraic invariant of a dessin d'enfant is its monodromy group. If we have a dessin and we label the  $n$  edges with the numbers  $1, 2, \dots, n$ , we can associate the dessin with a pair of permutations  $\sigma_0, \sigma_1 \in S_n$ , the symmetric group such that the cycles of  $\sigma_0$  correspond to the cyclic ordering (read counterclockwise) of the edges around the black vertices and the cycles of  $\sigma_1$  correspond to the ordering (read counterclockwise) of the edges around the white vertices. The *monodromy group* of a dessin with  $n$  edges is  $\langle \sigma_0, \sigma_1 \rangle$ , the group generated by  $\sigma_0, \sigma_1 \in S_n$ .

For example, see the dessin below, where we have a bicolored graph whose edges are labeled  $1, 2, \dots, 9$  inducing a pair of permutations  $\sigma_0 = (1, 2, 3)(4, 9, 8)(5, 6, 7), \sigma_1 = (3, 4, 5)(1, 9, 6)(2, 8, 7) \in S_9$  associated with the black and white vertices, respectively. The monodromy group of this dessin is isomorphic to  $C_3 \times C_3$ , where  $C_3$  is the cyclic group of size 3.



## Dessin on Triangular Billiards Surfaces

In this paper, we examine the dessin that gets drawn on a triangular rational billiard surface. Each angle of these triangles is a rational multiple of  $\pi$  [1]. To denote each triangle, we use

the triple notation  $(p_0, p_1, p_2)$  where  $p_0, p_1, p_2 \in \mathbb{N}$ . To retrieve the angles of a triangle, use  $(\theta_0, \theta_1, \theta_2) = (\frac{p_0\pi}{n}, \frac{p_1\pi}{n}, \frac{p_2\pi}{n})$  where  $n = p_0 + p_1 + p_2$  (e.g. the triple  $(1, 1, 1)$  represents the triangle with angles  $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$ , or the equilateral triangle). Note that if  $\gcd(p_0, p_1, p_2) = k \neq 1$ , the triple can be reduced to  $(p_0, p_1, p_2) = (\frac{p_0}{k}, \frac{p_1}{k}, \frac{p_2}{k})$ . Therefore, we will only consider triples in their most reduced form, writing  $T(p_0, p_1, p_2)$  for the rational triangle and  $X(p_0, p_1, p_2)$  for its corresponding billiards surface.

The number of copies of  $T(p_0, p_1, p_2)$  needed to form  $X(p_0, p_1, p_2)$  is  $2n$  [1]. A natural way of drawing the dessin on this billiard surface is drawing the black vertices on the copies with the same orientation as the original triangle and the white vertices on the copies with the opposite orientation. The edges are then drawn between two vertices if their corresponding triangles share an edge. We write  $D(p_0, p_1, p_2)$  for the dessin drawn on the billiards surface  $X(p_0, p_1, p_2)$ . From these conditions for drawing the dessin on a triangular rational billiard surface, some basic properties can be stated:

1.  $D(p_0, p_1, p_2)$  will have  $n$  white vertices and  $n$  black vertices.
2. Each vertex in  $D(p_0, p_1, p_2)$  has degree 3.
3.  $D(p_0, p_1, p_2)$  will have  $3n$  edges.

Write each of the permutations  $\sigma_0$  and  $\sigma_1$  as a product of disjoint cycles. The cycles of  $\sigma_0$  correspond to the cyclic ordering of the black vertices while the cycles of  $\sigma_1$  correspond to the cyclic ordering of the white vertices. One can easily derive the following properties of  $\sigma_0$  and  $\sigma_1$ :

1.  $|\langle \sigma_0 \rangle| = |\langle \sigma_1 \rangle| = 3$
2. The permutations  $\sigma_0$  and  $\sigma_1$  can each be written as a product of disjoint  $n$  3-cycles.

## Results

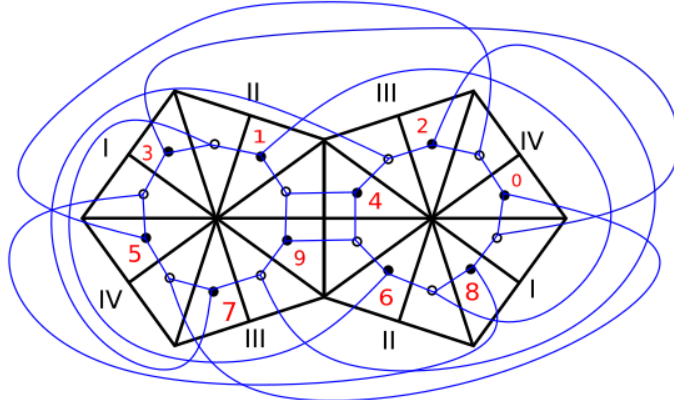
The main goal of this section is the classification of all monodromy groups corresponding to triangular billiards surfaces, as stated in the following theorem.

**Theorem 1.** *Fix  $p_0, p_1, p_2 \in \mathbb{N}$  with  $\gcd(p_0, p_1, p_2) = 1$ . Let  $G = \langle \sigma_0, \sigma_1 \rangle$  be the monodromy group of the dessin  $D(p_0, p_1, p_2)$  drawn on the triangular billiards surface  $X(p_0, p_1, p_2)$ . Setting  $N = \langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle$  and  $H = \langle \sigma_0 \rangle$ , we have  $G = N \rtimes H$ . Furthermore, if  $n = p_0 + p_1 + p_2$  and  $\alpha = \gcd(n, p_0 p_1 - p_2^2)$ , then*

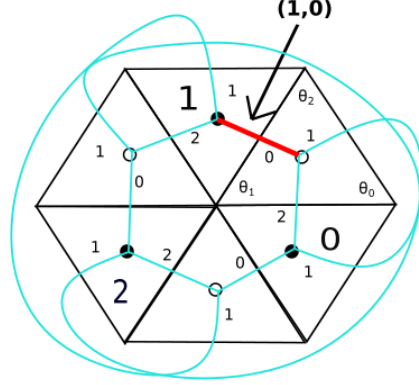
$$G \cong (C_n \times C_{\frac{n}{\alpha}}) \rtimes C_3.$$

## Notation

The billiards surface of a triangle is generated using reflections. Fix a triangle with a black vertex and label its vertex 0. Up to translation, half of the triangles (which have black vertices) of the billiards surface will be rotations of this fixed triangle while the other half (which have white vertices) will be reflections of those rotations. A useful way of labeling an edge on the dessin is in reference to the unique black vertex it is connected to. The black vertex labeled  $m$  will be associated with the triangle rotated  $\frac{2m\pi}{n}$  radians counter clockwise from the starting triangle. An example of this system for labeling the vertices is provided below:



Label each side of a triangle  $s_i$  where  $\theta_i$  is the angle opposite of that side. Label each edge  $(m, i)$  where  $m$  is the black vertex incident to the edge and  $s_i$  is the side of the triangle through which the edge passes. For this notation system, it is important that the angles  $(\theta_0, \theta_1, \theta_2)$  of a triangle with a black vertex are ordered counterclockwise.



The monodromy group acts on the set of edges according to the following formulas:

$$\begin{aligned}\sigma_0[(m, i)] &= \begin{cases} (m, 1) & \text{if } i = 0 \\ (m, 2) & \text{if } i = 1 \\ (m, 0) & \text{if } i = 2 \end{cases} \\ \sigma_1[(m, i)] &= \begin{cases} (m - p_1, 2) & \text{if } i = 0 \\ (m - p_2, 0) & \text{if } i = 1 \\ (m - p_0, 1) & \text{if } i = 2 \end{cases} \end{aligned} \quad (1)$$

**Lemma 1.** *The permutations  $\sigma_0\sigma_1$  and  $\sigma_1\sigma_0$  commute.*

*Proof.* Using (1), the permutations  $\sigma_0\sigma_1$  and  $\sigma_1\sigma_0$  can be computed to be the following:

$$\begin{aligned}\sigma_0\sigma_1[(m, i)] &= \begin{cases} (m - p_1, 0) & \text{if } i = 0 \\ (m - p_2, 1) & \text{if } i = 1 \\ (m - p_0, 2) & \text{if } i = 2 \end{cases} \\ \sigma_1\sigma_0[(m, i)] &= \begin{cases} (m - p_2, 0) & \text{if } i = 0 \\ (m - p_0, 1) & \text{if } i = 1 \\ (m - p_1, 2) & \text{if } i = 2 \end{cases} \end{aligned} \quad (2)$$

Furthermore, we compute

$$\begin{aligned}(\sigma_0\sigma_1)(\sigma_1\sigma_0)[(m, i)] &= \begin{cases} (m - p_2 - p_1, 0) & \text{if } i = 0 \\ (m - p_0 - p_2, 1) & \text{if } i = 1 \\ (m - p_1 - p_0, 2) & \text{if } i = 2 \end{cases} = \begin{cases} (m + p_0, 0) & \text{if } i = 0 \\ (m + p_1, 1) & \text{if } i = 1 \\ (m + p_2, 2) & \text{if } i = 2 \end{cases} \\ (\sigma_1\sigma_0)(\sigma_0\sigma_1)[(m, i)] &= \begin{cases} (m - p_1 - p_2, 0) & \text{if } i = 0 \\ (m - p_2 - p_0, 1) & \text{if } i = 1 \\ (m - p_0 - p_1, 2) & \text{if } i = 2 \end{cases} = \begin{cases} (m + p_0, 0) & \text{if } i = 0 \\ (m + p_1, 1) & \text{if } i = 1 \\ (m + p_2, 2) & \text{if } i = 2 \end{cases} \end{aligned}$$

demonstrating that  $\sigma_0\sigma_1$  and  $\sigma_1\sigma_0$  commute.  $\square$

**Lemma 2.** *Given a triple  $(p_0, p_1, p_2)$  and  $n = p_0 + p_1 + p_2$ , then  $p_0p_1 - p_2^2 \equiv p_0p_2 - p_1^2 \equiv p_1p_2 - p_0^2 \pmod{n}$ .*

*Proof.* Since  $p_2 \equiv -p_0 - p_1 \pmod{n}$ ,

$$\begin{aligned} p_0 p_1 - p_2^2 &\equiv p_0 p_1 - (-p_0 - p_1)^2 = p_0 p_1 - 2p_0 p_1 - p_0^2 - p_1^2 = \\ &-p_0 p_1 - p_0^2 - p_1^2 = p_0(-p_1 - p_0) - p_1^2 \equiv p_0 p_2 - p_1^2 \pmod{n}. \end{aligned}$$

The other congruence is proven similarly.  $\square$

**Lemma 3.** *Let  $\alpha = \gcd(n, p_0 p_1 - p_2^2)$ . Then  $|\langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle| = \frac{n^2}{\alpha}$ .*

*Proof.* We define a homomorphism  $\varphi : \langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle \rightarrow (\mathbb{Z}/n\mathbb{Z})^3$ , where  $(\mathbb{Z}/n\mathbb{Z})^3$  is a  $\mathbb{Z}/n\mathbb{Z}$  module viewed as a group, by  $\varphi(\sigma_0 \sigma_1) \mapsto v_1$  and  $\varphi(\sigma_1 \sigma_0) \mapsto v_2$  where

$$v_1 = - \begin{bmatrix} p_1 \\ p_2 \\ p_0 \end{bmatrix}, \quad v_2 = - \begin{bmatrix} p_2 \\ p_0 \\ p_1 \end{bmatrix}.$$

We claim that  $\varphi$  is injective. To show that this fact, suppose that two elements in the codomain are equal. That is, for  $(\sigma_0 \sigma_1)^{k_1} (\sigma_1 \sigma_0)^{k_2}$  and  $(\sigma_0 \sigma_1)^{l_1} (\sigma_1 \sigma_0)^{l_2}$  in the domain for some  $k_1, k_2, l_1, l_2 \in \mathbb{Z}/n\mathbb{Z}$

$$k_1 v_1 + k_2 v_2 = l_1 v_1 + l_2 v_2.$$

This gives the list of equalities

$$\begin{aligned} k_1(-p_1) + k_2(-p_2) &= l_1(-p_1) + l_2(-p_2) \\ k_1(-p_2) + k_2(-p_0) &= l_1(-p_2) + l_2(-p_0) \\ k_1(-p_0) + k_2(-p_1) &= l_1(-p_0) + l_2(-p_1). \end{aligned}$$

Using (2), one confirms that

$$(\sigma_0 \sigma_1)^{k_1} (\sigma_1 \sigma_0)^{k_2} = (\sigma_0 \sigma_1)^{l_1} (\sigma_1 \sigma_0)^{l_2}$$

which shows that  $\varphi$  is injective. Thus,  $\langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle$  maps bijectively onto  $\text{Span}\{v_1, v_2\}$ .

Therefore, finding the order of  $\text{Span}\{v_1, v_2\}$  will also give the order of  $\langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle$ . We will proceed using row reduction on  $[-v_1 -v_2]$ . First, recall that  $\gcd(p_0, p_1, p_2) = 1$  and  $p_0 + p_1 + p_2 = n$ . This implies that  $\gcd(p_1, p_2, n) = 1$ , since a number that divides  $p_1, p_2$ , and  $n$  will also divide  $p_0$ . Thus there exist  $s, t, u \in \mathbb{Z}$  such that  $sp_1 + tp_2 + un = 1$  and as a consequence  $sp_1 + tp_2 \equiv 1 \pmod{n}$ .

Observe

$$\begin{bmatrix} 1 & 0 & 0 \\ -(sp_2 + tp_0) & 1 & 0 \\ -(sp_0 + tp_1) & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_0 \\ p_0 & p_1 \end{bmatrix} \begin{bmatrix} s & -p_2 \\ t & p_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & p_0 p_1 - p_2^2 \\ 0 & p_1^2 - p_0 p_2 \end{bmatrix}$$

where

$$\det \left( \begin{bmatrix} 1 & 0 & 0 \\ -(sp_2 + tp_0) & 1 & 0 \\ -(sp_0 + tp_1) & 0 & 1 \end{bmatrix} \right) = \det \left( \begin{bmatrix} s & -p_2 \\ t & p_1 \end{bmatrix} \right) = 1.$$

Recall by Lemma 2 that  $p_0 p_2 - p_1^2 \equiv p_0 p_1 - p_2^2 \pmod{n}$ . Then this row reduced matrix is equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & p_0 p_1 - p_2^2 \\ 0 & -(p_0 p_1 - p_2^2) \end{bmatrix}.$$

Thus,  $|\text{Span}\{v_1, v_2\}| = n \cdot \frac{n}{\alpha}$  where  $\alpha = \gcd(n, p_0 p_1 - p_2^2)$ . As a result,  $|\langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle| = \frac{n^2}{\alpha}$ .  $\square$

**Lemma 4.**  *$N$  is a normal subgroup of  $G$ .*

*Proof.* Since  $N = \langle \sigma_0 \sigma_1, \sigma_1 \sigma_0 \rangle$  and  $G = \langle \sigma_0, \sigma_1 \rangle$ , proving  $N \triangleleft G$  is equivalent to proving the following four statements:

- $\sigma_0(\sigma_1 \sigma_0) \sigma_0^{-1} \in N$
- $\sigma_1(\sigma_0 \sigma_1) \sigma_1^{-1} \in N$

- $\sigma_0(\sigma_0\sigma_1)\sigma_0^{-1} \in N$
- $\sigma_1(\sigma_1\sigma_0)\sigma_1^{-1} \in N$

The first two statements are easy to show. To prove the third statement, observe that  $\sigma_0^{-1} = \sigma_0^2$  and  $\sigma_1^{-1} = \sigma_1^2$ . Thus,

$$\sigma_0(\sigma_0\sigma_1)\sigma_0^{-1} = \sigma_0^2\sigma_1\sigma_0^2 = (\sigma_0^2\sigma_1^2)(\sigma_1^2\sigma_0^2) = (\sigma_1\sigma_0)^{-1}(\sigma_0\sigma_1)^{-1} \in N.$$

The proof of the fourth statement is similar. □

**Lemma 5.**  $N \cap H = \{id\}$

*Proof.* Recall that  $N = \langle \sigma_0\sigma_1, \sigma_1\sigma_0 \rangle$  and  $H = \langle \sigma_0 \rangle$ . Suppose the intersection of these groups is not trivial. Then there is an element in  $N$  that is equal to  $\sigma_0$  or  $\sigma_0^{-1}$  which implies that  $H \subset N$ . Then the following should also be true:

$$\sigma_1\sigma_0\sigma_0^{-1} = \sigma_1 \in N$$

Then the group  $N = \langle \sigma_0, \sigma_1 \rangle = G$ . Since  $N$  is abelian by Lemma 1, the elements  $\sigma_0$  and  $\sigma_1$  must commute. By examining the proof of Lemma 1, we observe that  $\sigma_0$  and  $\sigma_1$  commute only when  $p_0 \equiv p_1 \equiv p_2 \pmod{n}$ . This only occurs when  $p_0 = p_1 = p_2 = 1$ . In this case, one observes that  $N = \langle \sigma_0\sigma_1, \sigma_1\sigma_0 \rangle = \langle \sigma_0\sigma_1 \rangle$  since  $\sigma_0\sigma_1 = \sigma_1\sigma_0$ . Therefore, either  $\sigma_0\sigma_1 = \sigma_0$  or  $(\sigma_0\sigma_1)^2 = \sigma_0$  since  $|\sigma_0\sigma_1| = 3$ . If  $\sigma_0\sigma_1 = \sigma_0$  then  $\sigma_1 = id$ , a contradiction. If  $(\sigma_0\sigma_1)^2 = \sigma_0$ , then  $\sigma_0 = \sigma_1$  which is also a contradiction. Hence,  $N \cap H = \{id\}$ . □

**Lemma 6.**  $NH = G$

*Proof.* Recall that  $N = \langle \sigma_0\sigma_1, \sigma_1\sigma_0 \rangle$ ,  $H = \langle \sigma_0 \rangle$ , and  $G = \langle \sigma_0, \sigma_1 \rangle$ . To show that  $\sigma_0 \in NH$ , choose  $n = id \in N$  and  $h = \sigma_0 \in H$ . Then  $nh = \sigma_0 \in NH$ . To show that  $\sigma_1 \in NH$ , choose  $n = \sigma_1\sigma_0 \in N$  and  $h = \sigma_0^{-1} \in H$ . Then  $nh = \sigma_1 \in NH$ . The generators of  $G$  are in  $NH$ , so  $NH = G$ . □

**Proof of Theorem 1.**

The group  $G$  is a semi direct product of subgroups  $N$  and  $H$  if and only if the three conditions are true:

- I.  $N \triangleleft G$
- II.  $N \cap H = \{id\}$
- III.  $NH = G$

Conditions I, II, and III are satisfied by Lemmas 4, 5, and 6 respectively. Therefore,  $G$  is a semi direct product of subgroups  $N$  and  $H$ . □

## Future Directions

In future research, we plan on investigating the monodromy groups of dessins d'enfant associated to rational billiards surfaces created by polygons with more than three sides.

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